Mathematical Methods 2
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Orthogonal polynomials
Lecture - 48
Legendre polynomials

So, in this lecture we move on to the next class of orthogonal polynomials namely the Legendre Polynomials and set up the scene to work out its properties ok.

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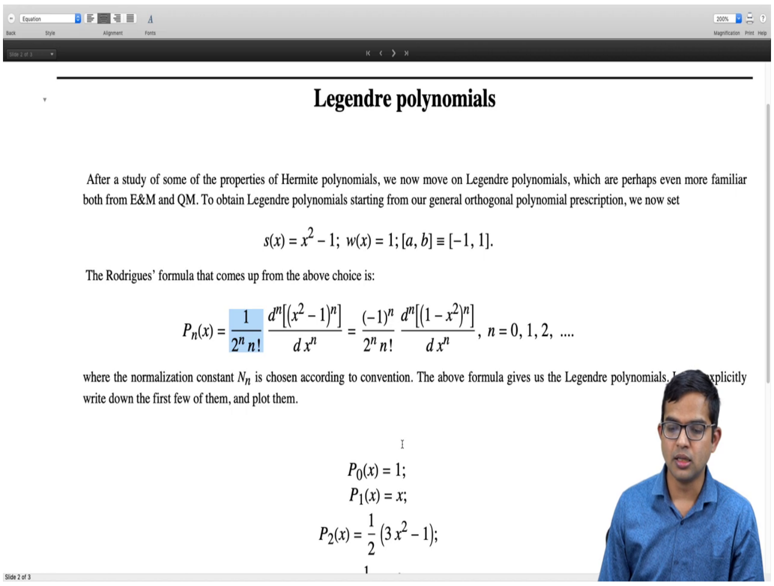

So, Legendre polynomials are familiar objects, they are seen in applications in E and M and quantum mechanics. So, its, but most likely these have been encountered starting from the differential equation and then probably some of the properties are familiar to you. But let us get them starting from our prescription.

So, we have this function s of x is equal to x squared minus 1 the weight function is taken to be 1 and our interval is restricted to lie between minus 1 and plus 1 right.

So, both ends are finite, so we have seen how it is always possible to restrict it to minus 1 to plus 1. So, the Rodrigues formula that comes up when you plug in these you know s and w and the interval in this manner is simply 1 over 2 to the n n factorial in the denominator and the nth derivative of the function x squared minus 1 the whole power n.

And it is often convenient to pull out a minus 1 from each of these factors x squared minus 1 and then write it as minus 1 to the n nth derivative of the function 1 minus x squared to the n instead of x squared minus 1 to the n.

And so, this is sometimes a convenient way of basically rewriting the same thing and n can take all integer values starting from 0 0 1 2 all the way you know forwards and the normalization constant here is chosen according to convenience. So, we will discuss this again as we go along. So, if we use this, let us look at the first few of them.

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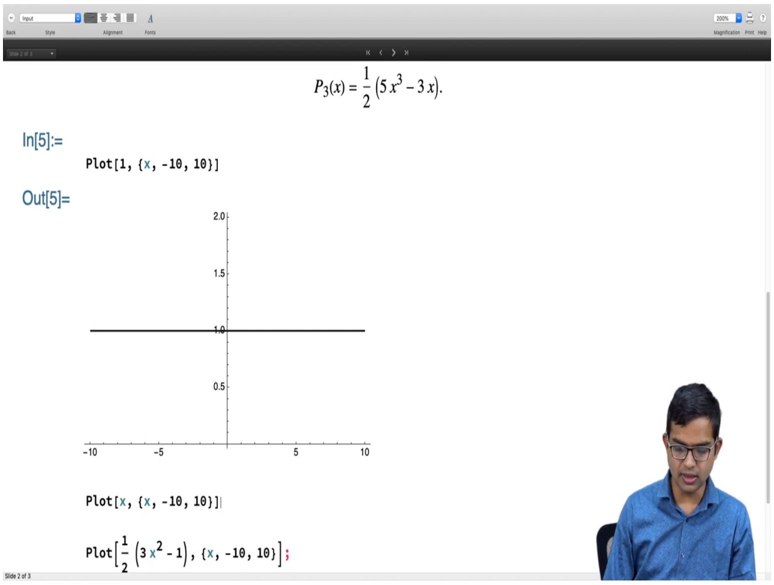

So, the normalization you know is set in such a way that P naught of 1 is 1. So, if you see the first few polynomials according to this if I explicitly write down P naught of x is equal to 1, P 1 of x is just x you should just you know plug in into this equation and check that indeed P 1 of x is 1 is x, P 2 of x is one half times 3 x squared minus 1 P 3 of x is one half times P 5 x cube minus 3 x.

So, these polynomials you know are quite similar to Hermite polynomials in the sense that they all have exact parity. And the alternate between oddness and evenness as we can see by looking at the plot of these functions the first one is just completely trivial it is just a constant P naught of x is 1.

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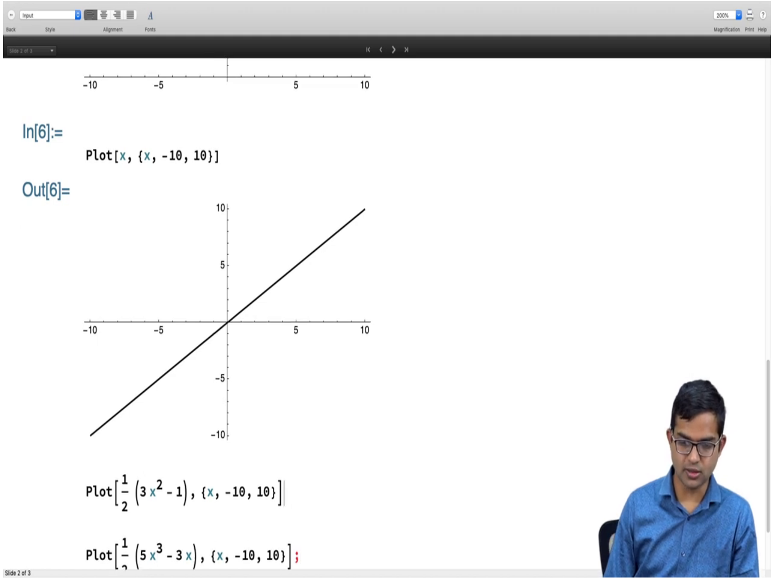

If I look at the second of these that is once again you know a simple function is just P 1 of x is x it is an odd function.

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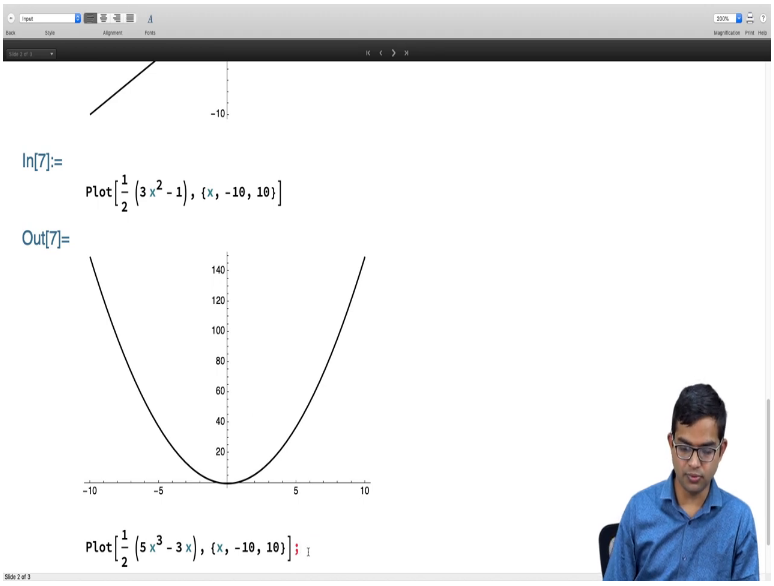

And then if I go to the next one it is a quadratic function which is even. So, it is 3 x squared minus 1 over 2 and then I have one more in this list and it will go on. So, it has even parity and the fourth one here has once again odd parity.

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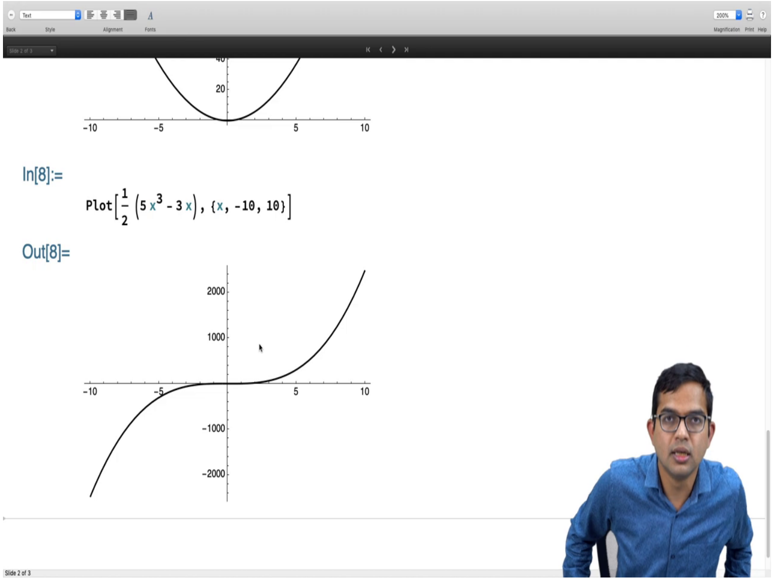

So, it is 5 x cube minus 3 x the whole thing divided by 2 right. So, there is a normalization that does not do anything as far as the parity is concerned. So, all the members of the sequence have definite parity and they alternate in you know oddness and evenness very similar to Hermite polynomials.

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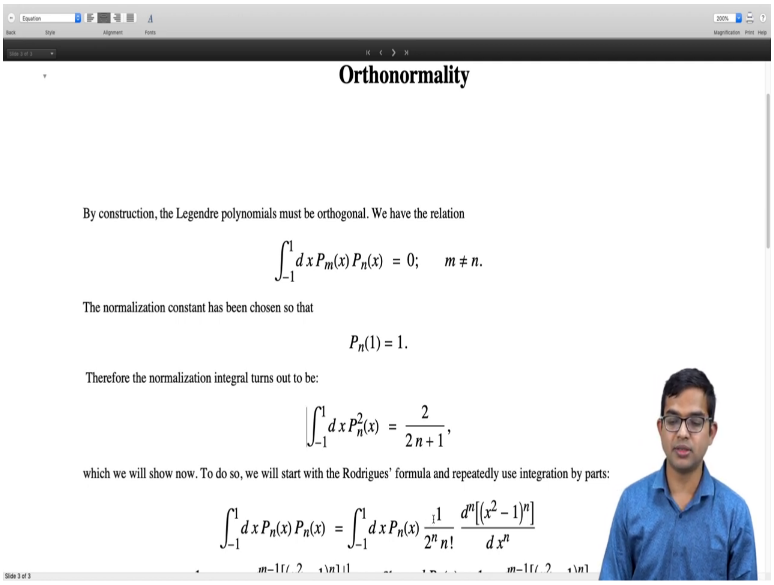

So, let us look at some more properties. So, the most important one is the fundamental property on which this entire structure is built is the ortho orthogonality right. So, the orthogonality is a given right. So, you should take any two polynomials in the sequence, you know put them together and the weight function is just a weight function is just 1 and s of x does not matter as far as this integral is concerned it is minus 1 to 1 dx P m of x times P n of x must be 0 whenever m is not equal to n right.

So, the normalization constant has been chosen so that you fix P n of 1 equal to 1 right as we have seen the examples we considered right. So, you set P n of 1 to be 1 for all m and so, if you do that it gives us the normalization constant that I mean we will explicitly show what the normalization integral is.

So, let us derive this result right. So, starting from the Rodrigues formula and using integration by parts right. So, let us explicitly check this. So, I mean I have given you the answer perhaps this is also something that is familiar to you and there are in fact, multiple ways of working this result.

So, I have one way right and I am going to show you that result, but if you want to pause the video and try to find your own way of deriving this result, it would be certainly encouraged.

So, let us show you my approach. So, what I do is, start with this integral minus 1 to 1 dx P n of x squared I write it as P n of x as P n of x and for one of these P n of x is I replace it by the Rodrigues formula. So, I have 1 over 2 to the n n factorial times the nth derivative of x squared minus 1 to the n.

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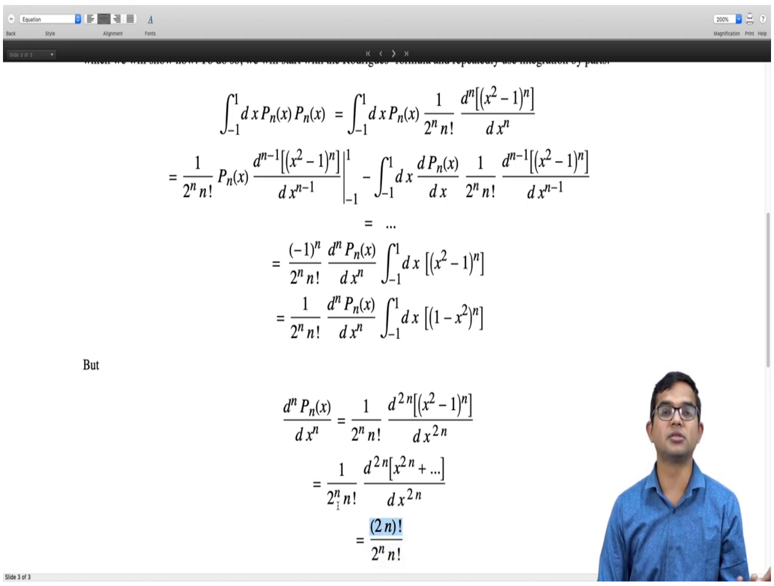

And now I just carry out an integration by parts. So, I have u dv I write it as u v minus v d u u is just this guy and then v is the n minus oneth derivative of this function evaluated between at the points plus 1 and at minus 1 minus integral minus 1 to 1 d x now I have to take a derivative of this.

So, I have to do v d u. So, d d by dx of P n of x times the factor remains as it is 1 over 2 to the n n factorial times the n minus oneth derivative of x to the x squared minus 1 the whole power n.

And now we argue that this boundary term is in fact, 0. So, the reason is, you are taking the n minus 1th derivative of this x squared minus 1 to the n right. So, whenever you take a derivative with respect to x, what is it going to give? Suppose I do it just once then I have x squared minus 1 to the n.

So, that gives me n times x squared minus 1 the whole power n minus 1 times some stuff, but the key point is that every one of the terms when I take a derivative with respect to this only n minus 1 times will have at least one factor of x squared minus 1.

So, if you put x equal to 1 or if you put x equal to minus 1 term by term every one of them is going to vanish. So, indeed at each of the boundaries separately this term vanishes and therefore, you can actually ignore this boundary term and you are just left with this integral which again we integrate by parts. So, once again you treat this to be your u and this is your d v.

So, you should then you will have to get again a boundary term of this kind and then minus now the second order derivative here and then in place of n minus 1 here you will get n minus 2. So, you know one of these derivatives' order keeps on increasing while the other one keeps decreasing and so, every time you get a minus sign. So, which comes from you know taking a derivative with respect to x squared minus 1.

So, you have, so let us check this again. So, you have u d v, well I mean it comes from the fact that you have u v minus v d u right. So, this is just v and this is just d u, but. So, let us check this again, so u d v is u v minus v d u. So, the minus sign, so there is going to be one more minus sign. So, it just comes from the negative sign from the integration by parts, so you have a minus sign once.

Then you get a when you do it again you get another minus sign the third time you will get a minus sign and so on, so which is conveniently written as just minus 1 to the n. You keep on doing this repeatedly there is going to be a point at which you are going to get the nth derivative of this polynomial P n of x.

And this is going to become all the derivatives here, get exhausted and you are left with just x squared minus 1 the whole power n. And so, notice here I have pulled out this nth derivative outside of the integral.

The reason I can do that is because P n of x is a polynomial of degree n and if I differentiate it with respect to x n times all the dependence on x is gone and I am just going to be left with a constant which I can pull out.

Therefore, it can come out of the integral and then I am left with just this very simple looking integral minus 1 to 1 dx x squared minus 1 the whole power n right. So, it is convenient to bring this minus 1 back in and write this as.

So, this normalization integral as 1 over 2 to the n n factorial times the nth derivative of this P n of x times minus 1 to 1 dx times 1 minus x squared the whole power n. So, these three factors right. So, 1 over this is just a constant and then there is this nth derivative which we have to work out and there is also this integral which we have to work out. So, we will proceed to work these two quantities out, you know in steps. So, let us first look at this quantity.

So, what is this quantity? If I want to do the nth derivative of P n of x which is nothing, but 1 over, so I put in the Rodrigues formula for P n of x. So, which gives me 1 over 2 to the n n factorial, now the Rodrigues formula already brings you an nth derivative and there are again further n derivatives which need to be taken. So, I get a 2 nth derivative with respect to x of this function x squared minus 1 the whole power n.

Now, this is something which is easy to evaluate because you know the stuff that is whose derivative is whose 2 nth derivative is being taken is a polynomial of degree 2 n. 2 n because you know you just expand. So, you will see that if you keep on differentiating there is only one term which will survive the derivative which is being taken to n times.

So, the highest order term is just a x to the 2 n plus there is other stuff which you do not even have to worry about because you are taking derivatives to n times.

So, all of them are going to be killed in this process of repeated differentiation. When you take a derivative of the first term once you get a factor of 2 n, the second time you are going to get 2 n times 2 n minus 1, the third time you are getting another factor of 2 n minus 2 and so on. So, you can convince yourself that in fact, all that this does is to give you this factor of 2 n the whole factorial.

And then of course, the denominator there is 2 n 2 to the n and n factorial. So, overall it's straightforward to show that in fact, the nth derivative of this polynomial P n of x is 2 n the whole factorial divided by 2 to the n times n factorial both of these factors come in the denominator. So, one of these we have already managed to work out. The other one is to work out this integral which we will again work out by integration with parts.

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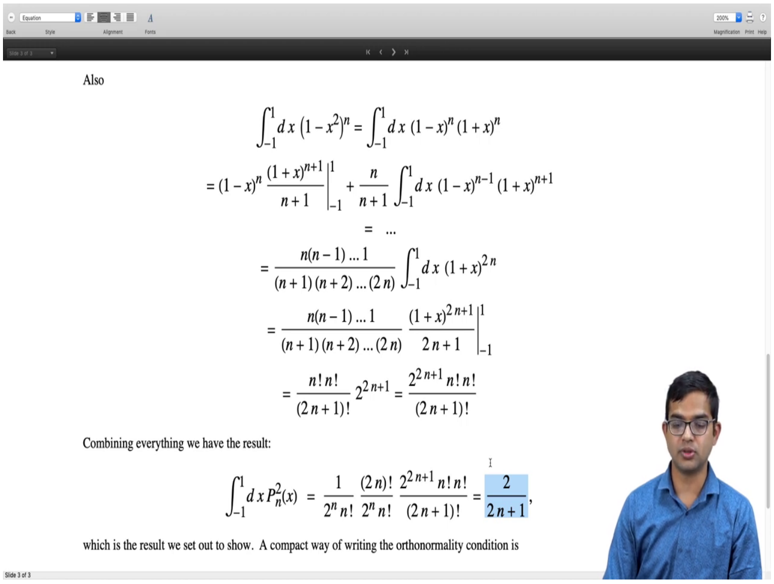

So, the quantity we want to work out is minus 1 to 1 d x 1 minus x squared the whole power n we write this as the integrand as 1 minus x to the n times 1 plus x to the n and then we proceed to do this by parts. So, we have u d v, so let us take 1 plus x to the n as you know times d x as this d v.

So, this is going to be 1 minus x to the whole power n remains as it is times v which will be 1 plus x the whole power n plus 1 divided by n plus 1 evaluated at minus 1 and plus 1. So, minus, but then there is another minus sign which comes about because you take a derivative of this quantity 1 minus x to the whole power n you do n times 1 minus x to the n minus 1 times minus 1.

So, that becomes a plus n and then you have a denominator n plus 1 that comes from here, so this 1 plus x the whole power n plus 1 right. So, you see that in one of these terms there is a reduction in the power and in the other one there is an increase in the power.

Now we argue once again that this boundary term will go to 0 the reason is you have these factors of 1 minus x and factors of 1 plus x. So, 1 of these is going to go to 0 at plus 1 and the other is going to make it go to 0 at minus 1. So, together for sure this quantity is going to be 0 right and we can repeat this procedure repeatedly.

So, each time this is going to be a plus sign right because 1 minus x the whole power n minus 1 will always give you an extra minus sign that will go along with this and give you a plus sign.

So, you can convince yourself that every time you get this factor of starting from n you get n into n minus 1 into n minus 2 so on all the way down to 1 and the denominator you keep getting factors which are you know one more than the previous. So, n plus 1 is the first one and the next time you are going to get n plus 2 then you are going to get n plus 3 so on all the way up to 2 n right.

So, in this process 1 minus x to the n minus 1 will become 1 minus x to the n minus 2 n minus 3 so on and then there comes a point where it is this factor is entirely gone you are left with just a factor of 1. And then on the other hand 1 plus x you are going to get more and more factors of 1 plus x until it becomes 1 plus x to the whole power 2 n. So, this is the only integral which we have to evaluate and there is this factor outside n factorial in the numerator and in the denominator this product of n plus 1 n plus 2 so, on all the way up to 2 n.

Now, this integral is actually straightforward to evaluate. So, you simply write down 1 plus x to the whole power 2 n plus 1 divided by 2 n plus 1 evaluated at minus 1 and plus 1 and you immediately see that when you put x equal to minus 1 you get a 0 right. So, the lower end does not matter it's only the upper end that you have to keep track of and there you see that you get 1 plus 1 to 2 to the 2 n plus 1 times.

So, this this is n factorial in the numerator if you know include another n factorial and put another n factorial in the denominator multiply the numerator and denominator by n factorial, you can conveniently rewrite the denominator tagging along all these extra stuff that you have put in 1 into 2 into 3 all the way up to n and simply write it as 2 n plus 1 factorial. And so, in place of just 1 n factor in the numerator you have you know a pair of them.

So, n factorial times n factorial divided by 2 n plus 1 the whole factorial times the only contribution here comes from the upper end. So, we have seen that that is just 2 to the 2 n plus 1. So, we have this compact expression for this entire thing which is just 2 to the 2 n plus 1 times n factorial times n factorial the whole thing divided by 2 n plus 1 factorial. So, combining all these three factors we have to multiply these three factors to work out this normalization integral.

So, first of all we have 1 over 2 to the n times n factorial and then we show we saw that this nth derivative of this polynomial P n of x is just going to give us 2 n the whole factorial by 2 to the n n factorial here. So, that is the second of these and then there is a third of these. So, in place whatever is here we put it in here a number of calculations happen.

So, this n factorials will go to 2 to the 2 n will go and then 2 n factorial n to n plus 1 you will only have this 2 n plus 1 left and only the numerator has only 2, eventually giving us the familiar result. So, the normalization integral finally, is seen to be just 2 divided by 2 n plus 1.

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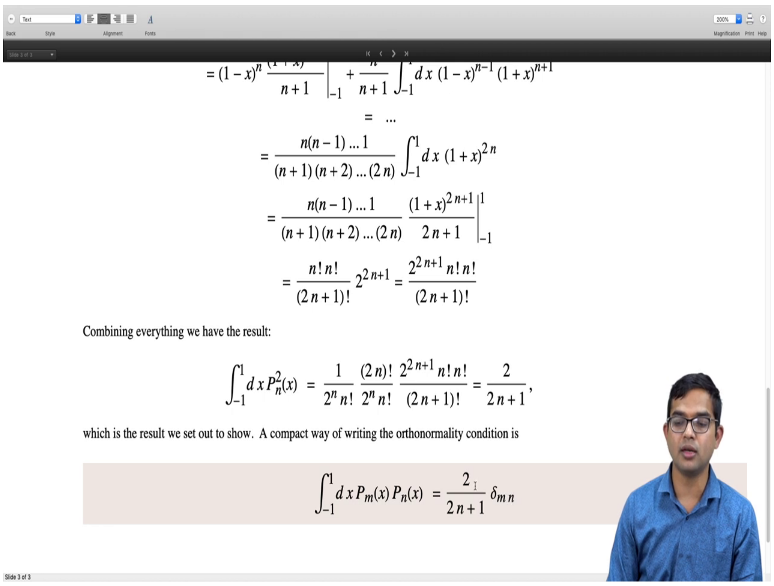

So, there is a compact way of you know I am adding these two together: the orthogonality condition which is there by design by construction and the normalization integral we have worked out. There is a way to combine these two and write it in this compact form as integral minus 1 to 1 dx P m of x times P n of x is equal to 2 divided by 2 n plus 1 times this chronicle delta.

So, this ensures that only when m equal to n the right hand side is nonzero and in fact, it is exactly equal to 2 divided by 2 n plus 1 and in all other cases when m is not equal to n for any 2 integers m and n m not equal to n these integral better be 0. Because that is the heart of the story, which is that all these different polynomials are orthogonal to each other. Ok that is all for this lecture.

Thank you.