Mathematical Methods 2
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Orthogonal polynomials
Lecture - 45
Differential equations satisfied by the orthogonal polynomials

So, we have seen how to construct a sequence of polynomials which are all orthogonal to each other with respect to some weight function in an interval. And we have seen how based on some very general arguments, it is possible to show that all such sets of polynomials must satisfy a three term recurrence relation.

So, in this lecture, we will see a general differential equation that such orthogonal polynomials would satisfy. And the arguments involved are quite clever about how to arrive at such a differential equation. So, it might be possible that some of you have already encountered such orthogonal polynomials actually starting from such a differential equation.

So, often in physics problems it is the differential equation which comes about from some real world application. And then we see that we know when we try to solve these differential equations using a power series approach like we did in some of the techniques we covered when we were discussing ODEs, and then the resulting solutions turn out to be some of these polynomials. And then we studied these polynomials often.

This is the route in which you know they were perhaps first studied and perhaps this is the root, and many discussions involving orthogonal polynomials also start from there. But in this the discussion we have followed has been to sort of look at it from an abstract perspective.

So, come up with a general approach to construct a set of polynomials, and then we are studying various general properties at this point. So, in this lecture, we will see how to make contact with the differential equations ok.

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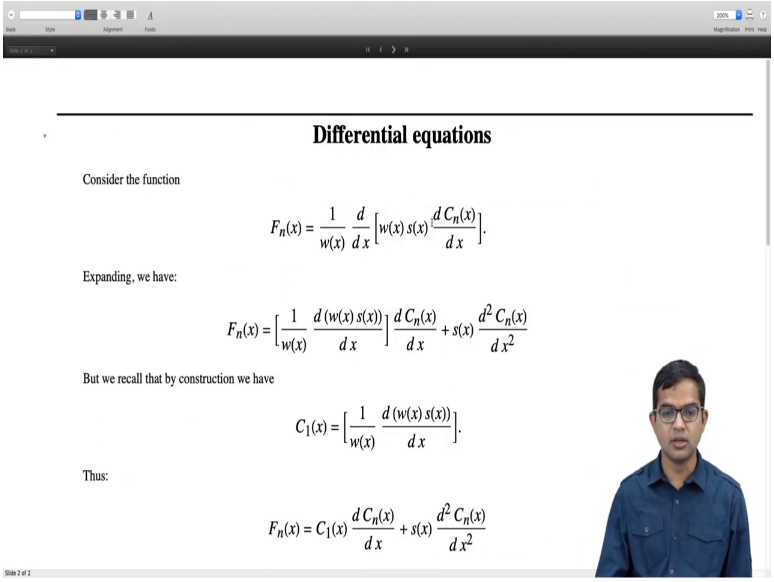

So, the trick is to consider this function F n of x. It looks somewhat like a contrived object, but you will see in some time you know why this turns out to be a useful quantity to study. So, what you do is you take your w, you take your s, and then multiply it by the derivative of this nth polynomial in your sequence, and then take the derivative of this object and then divide by w right, so that is how this function F n of x is defined right.

So, you will see in a moment that you know these are very interesting objects to study. It is a polynomial which we will argue for. So, ok let us not get ahead of the argument. So, you have this weird object. So, let us just do some algebra on it. So, we have you know we can always assign an index n to it because there is this index C n sitting on the right hand side.

So, now, we will just work out what this is right. So, if you expand this, we have you know think of this w times s as just some one function, and in this another function. So, you are taking the derivative of the product of two functions. And so you have 1 over w of x times the derivative of this product. And then you leave the other stuff as it is. You know you just leave d by dx of C n of x as it is.

And then the second term comes about when you treat this w of x times s of x as a constant and you pull it out, and so the w will cancel and then you are just left with s. And then when you take the derivative it becomes the second derivative with respect to C n and so with respect to x of the function C n of x. Now, but in fact this object here we should be able to identify it right.

So, from an earlier lecture, you can go and recap that is in fact nothing but the first of these polynomials C 1 of x, C naught of x is a trivial one. C 1 of x is identified to be this complicated stuff is actually nothing but the polynomial C 1 of x.

So, we can go ahead and rewrite this complicated construction as in fact nothing but F n of x is C 1 of x times the derivative of the nth polynomial C n of x plus s of x times the second derivative of this nth polynomial.

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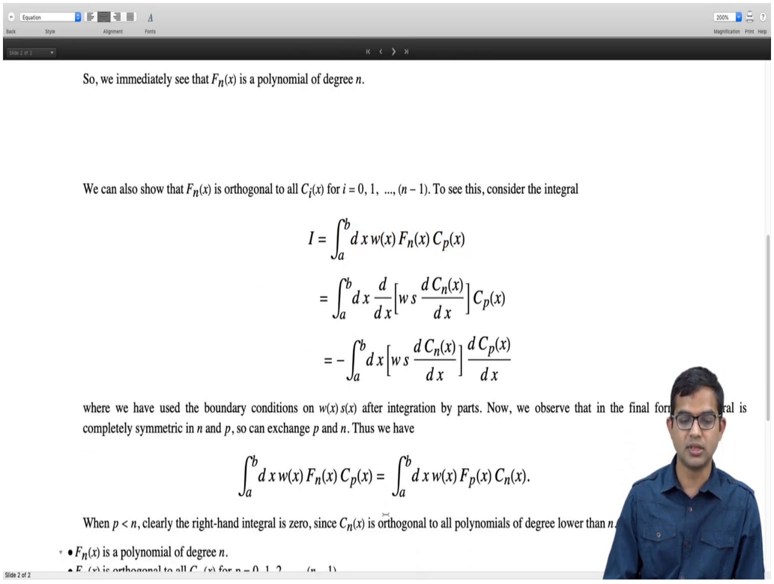

Now, from this relation, we can actually immediately say that this F n of x must be a polynomial of degree n. Why is that? So, you see C n of x is a polynomial of degree n. So, if you take a derivative with respect to x, it is a polynomial of degree n minus 1. And then you are multiplying by C 1 of x which is a polynomial of degree 1. So, necessarily the product of a degree 1 polynomial with a degree n minus 1 polynomial will give you a degree n polynomial.

Now, this is kind of inconsequential, but we can let us look at it. So, if you take the second derivative of an nth degree polynomial, it is going to give you an n minus 2 degree polynomial. So, now, we are when you multiply by s of x, you know the best it can do is to increase the order of this polynomial by 2 more right.

So, because s of x, the three choices we have x squared minus 1 which will make it degree n minus n minus 2 times 2 plus 2 which will become n; or if s of x is chosen to be x or if it is chosen to be 1, for sure it is this is the degree of this can never exceed n. So, overall this is indeed a polynomial because of the way in which you are tying together various polynomials.

And s of x is also either 1 or x squared minus 1 or x so that is also a polynomial. So, indeed this overall object is a polynomial, and it has degree n. So, now we will argue that it is not just any arbitrary polynomial, but it is in fact a familiar polynomial. To see this we will first show that in fact F n of x is orthogonal to all these C i of x, all these polynomials we have constructed where i goes from 0 to n minus 1.

To see this you have to consider this integral. So, integral from a to b dx the weight function w of x comes in then F n of x C p of x, where p you know can take any of these integer value starting from 0 and all the way up to n minus 1 and also n. So, p can in principle can also take the value n.

So, in order to evaluate this integral or you know take it a step further, we will replace this F n of x by this first expression we had for we have this 1 over w times a derivative. So, it is convenient to rewrite F n of x in this integral as 1 over x 1 over w times this derivative.

And this w will cancel with the w which is sitting here outside. So, you are left with this integral from a to b dx d by dx of this w times s times d by dx of C n of x. Then there is an overall C p of x which is sitting outside.

And now we integrate this by parts. So, d by dx, so you have you know u dv. So, it is going to become u v. So, the C p of x times w s d by d x that is the boundary term from a to b minus this stuff, but I mean immediately we can argue that the boundary term has to go to 0 because of the manner in which we have chosen w and s. w and s have been constructed explicitly in such a way that this product w s must be 0 at the boundaries.

So, since you have this exact product sitting here evaluated at a and b at each of the boundaries is equal to 0. So, the difference is also 0. And so we are left with just minus v du. So, in this case so u dv, so this is v. So, this stuff comes in as it is without the derivative and then you have to take a derivative of this and there is an overall minus sign right.

So, because you are doing u dv there is u v minus v du; v is here, v du is here, and there is one minus sign. So, this is just simply integration by parts right. So far it is just some manipulation of this integral and now comes a very clever argument. So, if you look at this, if you observe this integral on the right hand side carefully, you see that this is now symmetric in NNP.

Although what the integral that we started with you know you get an F n and you get a C p, but we have managed to show that this must be equal to this integral where if I exchange n and p, I get back the same integral right. And this is equal to another integral where apparently on the face of it, it is not symmetric.

But if this holds and if I can exchange n and p in this integral, then I should be able to exchange n and p in the other integral as well. And this is a great simplification. So, what we can argue is in fact integral a to b dx w of x F n of x C p of x is the same as integral a to b dx w of x F p of x C n of x. And this immediately leads to a very important consequence right.

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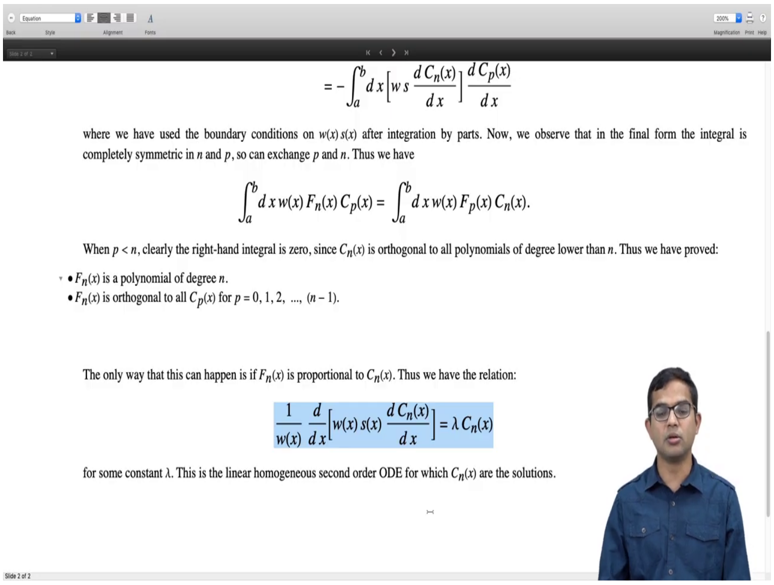

So, when p is less than n right, so clearly the right hand side is 0 right. So, if F p of x we have already argued is a polynomial of degree p right. So, and if p is less than n, so we know that C n of x is a polynomial from our box of polynomials. And C n of x is constructed in such a way that it is orthogonal to every polynomial whose degree is less than n.

And if F p is a polynomial of degree p, and p is less than n, we can immediately say that this right hand side is 0. And therefore, the left hand side is also 0 right. So, we have managed to show that whenever p is less than n in fact this integral is 0.

Therefore, we have managed to show that F n of x is a polynomial of degree n, and F n of x is orthogonal to all C p of x whenever p is less than n. So, p can take values from 0 all the way up to n minus 1, and then it is 0. So, the left hand side is 0.

So, what is this set of polynomials? So, we have managed to construct a sequence of polynomials F n of x which I have degree n, and these polynomials are orthogonal to all C p of x where t goes from 0 to n minus 1 right. So, this is a familiar object in fact, F n of x is nothing but C n of x except that you can have some arbitrary factor associated with it right

So, this is in fact how we constructed the polynomials C n of x. They have to be polynomials of degree n and have to be orthogonal to all lower order polynomials in that set, and that is exactly what is happening here. Therefore, in fact, this complicated looking object F n of x you know which we defined as 1 over w of x times d by dx of w times x times this derivative is in fact nothing but C n of x itself up to some arbitrary constant lambda.

So, in fact, this is the differential equation that we were looking for right. So, this is a linear second order homogeneous ordinary differential equation which if solved will actually recover for us the solutions that we started with right. So, this is often like we said at the beginning. In fact, the starting point is that we start with a differential equation of this kind, and then we try to find a power series solution for such a differential equation and then we discover these polynomials.

So, in this lecture, we have managed to go in the opposite direction, and show the differential equation for the sequence of polynomials we have constructed. We can actually work out the differential equation using the weight function and this function s of x, and everything is a nice combination.

And then they have argued that if you just do this operation. And then if you write it like this, this is the differential equation corresponding to the sequence of polynomials of interest ok. That is all for this lecture.

Thank you.