Mathematical Methods 2

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Orthogonal polynomials
Lecture - 44
Recursion relations satisfied by the orthogonal polynomials

So, we have come up with a prescription to build a sequence of polynomials which are orthogonal with respect to some weight function in some given interval and so, in this lecture we will see just based on these you know properties which come from first principles, it's possible to write down a general Recursion relation that orthogonal polynomials of this kind satisfy ok.

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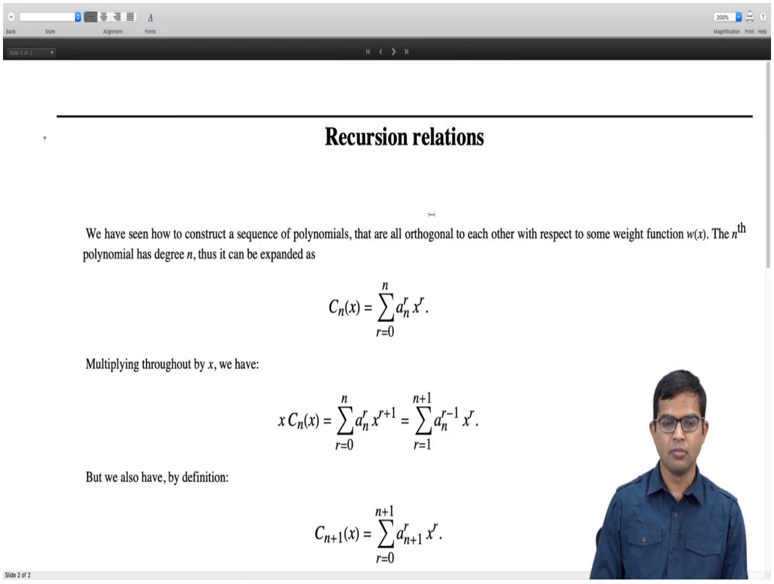

So, we start by writing down a generic nth degree polynomial right. So, our nth degree polynomial has a bunch of coefficients like this C n of x is equal to sum summation over r going from 0 to n, it's a degree n polynomial. So, an r x to the r. Now if you multiply throughout by x we get a polynomial of degree n plus 1 right.

So, we have x times C n of x is summation over r going from 0 to n an r these coefficients. So, this r here is superscript r is not to be not a power it's just there are these two parameters associated with these coefficients right. So, a n a subscript n superscript r and x to the power r plus 1 right.

So, it's convenient to write the same sum as in terms of r going from 1 to n plus 1 and an r minus 1 x to the power r right. So, all we have done is you know shifted r plus 1 to some other dummy variable. If I call it s, then r will become s minus 1 and s will go from 1 to n plus 1 and then there is no need to call it as you might as well just call it r right. So, that is how we get this. But C n plus 1 is another n plus 1 degree polynomial right.

So, I mean we have one n plus 1 degree polynomial and the idea is that we want to compare this n plus 1 th degree polynomial we constructed from the nth order polynomial by with respect to the n plus 1 degree polynomial itself which for which we have this expansion right. So, in terms of the same kind of notation, but now you see I have the superscript is r and the subscript is n plus 1 right. So, it corresponds to this n plus 1 polynomial and r will now go from 0 to n plus 1 and of course, you have an x to the r sitting here.

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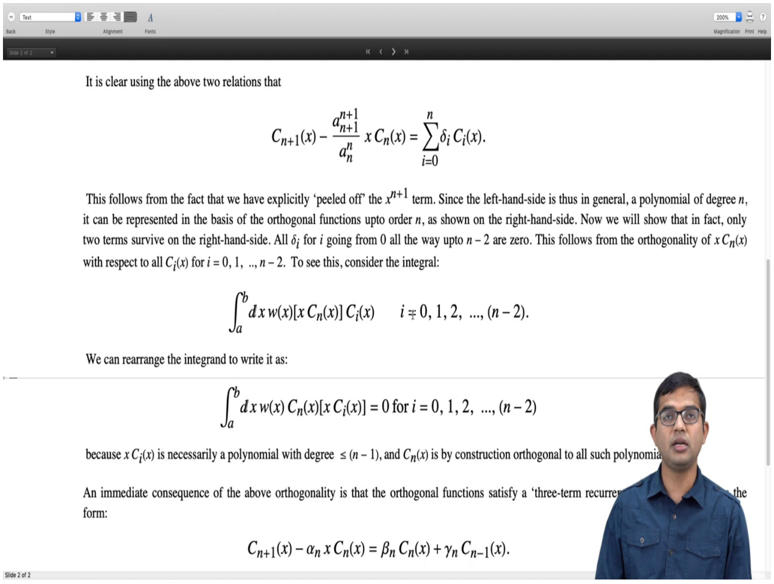

So, now if we compare these two polynomials. And in fact, we look at a particular combination of these two. So, what we do is, you take this polynomial C n plus 1 and then multiply this x times C n plus 1 of x by a suitably chosen factor. So, we have a n plus 1 n plus 1 divided by a n right and then times x times C n of x.

So, now comes a crucial argument right. So, what we are doing is, trying to peel off this highest order term right. So, what do we do? We know that C n plus 1 has this highest order term which is given by when you put r is equal to n plus 1 and so, that is going to be a n plus 1 n plus 1 and x to the n plus 1.

So, we want to come up with a way to remove this highest order term n plus 1th term. So, the highest order term here is an plus 1 and n plus 1 and the highest order term here is when r is equal to n plus 1. So, if r is equal n plus 1 we get a n n. So, if you take C n plus 1 of x and then subtract you know this factor times x plus x times n plus x times n of C n of x then indeed you are guaranteed that the result cannot be a polynomial of degree greater than n right.

So, I mean because we have explicitly the n plus 1th order term. So, there is no question of the n plus 1 order term surviving. So, in general you have a degree n polynomial and we know that these orthogonal polynomials form a basis for all polynomials of up to degree n. So, you can consider this nth degree polynomial on the right hand side and expand it in terms of these basis functions.

So, you can always find a set of coefficients delta i, where i runs all the way from 0 to n such that this summation delta i C i is indeed the polynomial that you want to extract its an nth degree polynomial all these coefficients delta i are for sure possible to find and in fact, it turns out that not all of these coefficients are going to be nonzero.

So, in fact, we will presently argue that at best there are two terms two coefficients here which are nonzero and they correspond to be i equal to n and i equal n minus 1. So, delta n and delta n minus 1 are you know are the only potentially nonzero coefficients in other words delta i for all i less than or equal to n minus 2 are for sure going to be 0 right. So, we are going to argue for that in a moment.

But the key point at this point is that in fact, we have managed to peel off the n plus oneth order term and it is a degree n polynomial which we can expand in this manner on the right hand side. So, in order to show that all these lower order terms are 0. So, let us consider this integral right. So, the result follows from orthogonality right.

So, we know that all these polynomials you know together they form a basis, but they are also orthogonal with respect to your weight function in the given interval right. So, to see that you know these delta i for i less than or equal to n minus 1 are 0. So, we consider this integral you know multiply of course, with w of x and then you take this x times C n of x and then you also put in C i of x and this I can go from 0 all the way up to n minus 2 you will see in a moment why we do this.

And so, i is restricted to lie between 0 and n minus 2 if you consider this integral now there is a you know way to relook at the same integral right. So, we recast this integral - it is the same integral, but we just shift this you know x to the right of C n of x and then group these two together x times C i of x.

And now C x times of C i of x for sure is a polynomial whose degree cannot be greater than n minus 1 because C i of x is a polynomial whose degree cannot be greater than n minus 2. So, if you multiply it by x this polynomial has a degree which is necessarily less than or equal to n minus 1, but C n of x is a polynomial which is degree of degree n and which is orthogonal to all polynomials whose degree is less than n with respect to this weight function in this interval right.

So, immediately this follows that this has to be 0 right. So, we see that this function x, x times C n of x cannot contain any term whose order is greater than n minus 2 right. So, an immediate consequence is that we get this three term recurrence. So, the key point is that you know that if any of these other C i's has to appear it has to appear only from here right only from the second term.

Because this first term will have no component coming from any C i of x, not just you know i less than or equal to n minus 2, but it's for any i other than n plus 1 this is not going to contribute because its already a an orthogonal polynomial with index n plus 1. And every other orthogonal polynomial whose degree is less than or equal to n minus 2, we have explicitly shown cannot appear here.

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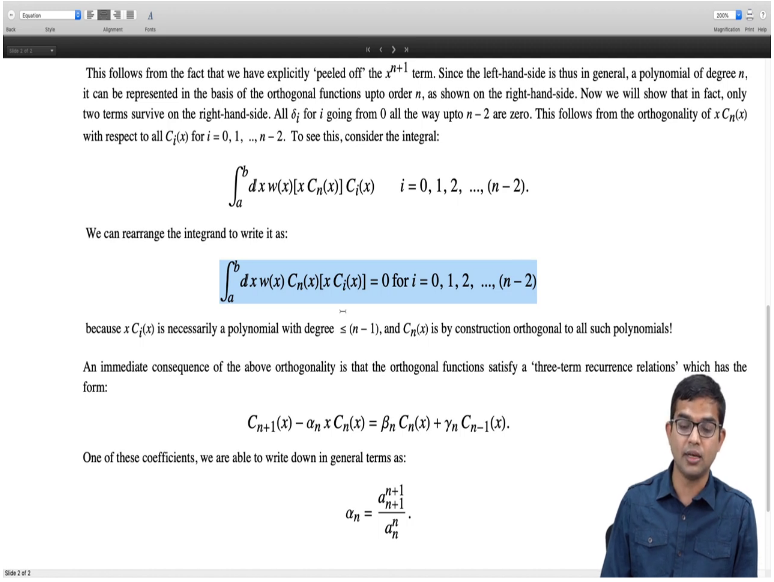

So, immediately we have this result: C n plus 1 of x minus some coefficients alpha n x C n of x is equal to beta n C n of x plus some other coefficient gamma n C n minus of n minus 1 of x. So, both these coefficients beta n and gamma n are to be determined and alpha n in fact, we already have it we have got it in this description already worked out.

So, alpha n is just the highest coefficient an plus 1 n plus 1 of this expansion for C n plus 1 divided by the highest coefficient in the expansion for highest coefficient term in the polynomial C n of x right.

So, basically we managed to do this clever argument and use this peeling off of operation to show that necessarily there is this way of combining this n plus 1-degree polynomial and nth degree polynomial in this precise manner to get this three term recurrence relation.

So, it turns out that specific polynomials satisfy a separate three term or some number of term recurrence relation which we will come to when we discuss individual polynomials, also derivatives involved and so on.

There are other interesting recurrence relations which can be worked out. But this is a very general result it holds for any of these different kinds of orthogonal polynomials which we will look at ok that is all for this lecture.

Thank you.