Mathematical Methods in Physics 2
Prof. Auditya Sharma
Department of Physics
Indian Institute of Science Education and Research, Bhopal

Orthogonal polynomials
Lecture – 41
Introduce orthogonal polynomials

So, with this lecture we begin a new topic. So, we will discuss orthogonal polynomials starting from a general perspective and then we will go on to you know some specific sets of polynomials and see how some of these general principles will apply in you know these specific cases, ok.

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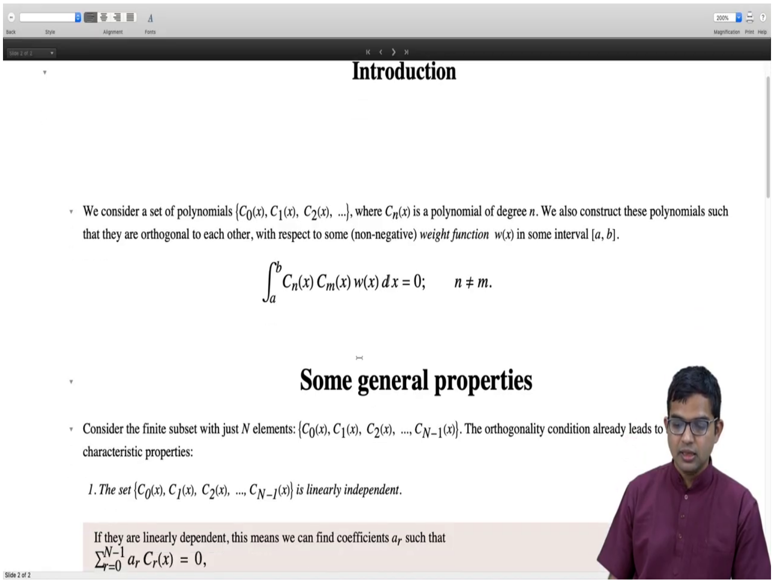

So, the idea is that we want to come up with a set of polynomials. Let us call them C naught of x, C 1 of x, C 2 of x so on; it is a an infinite set of polynomials a countable infinity of such polynomials where C n of x is going to be a polynomial of degree n, right. So, for every non-negative integer there is a corresponding polynomial with that degree.

And, such that all these polynomials you know have a nice relationship with each other, namely they are orthogonal to each other and with respect to a non-negative weight function which also needs to be defined and there is also a an interval which is taken to be between a and b, right. So, all these ingredients are essential to define this idea of orthogonality, right.

Orthogonality simply means that this integral from a to b, right that is the interval in which we have you know we are considering these are polynomials and with respect to this weight function. So, if you take this integral a to b C n of x C m of x weight function dx is going to be 0 whenever n is not equal to m. So, that is what is meant by the orthogonality of these polynomials.

And, the reason why we need this weight function is because you know we need the notion of an inner product, right. So, and you know often you may have a scenario, where a and b can go to minus infinity and plus infinity, and it is the weight function which is going to ensure that you know these integrals will converge and in a nice way.

So, in order to ensure convergence of such integrals and for the notion of an inner product to be available right so, we are going to start thinking of these polynomials as elements of a vector space, right. So, and we have seen how it is useful to be able to you know work with the notion of an inner product and so, these integrals can be thought of as inner products of vectors and for them to be well defined these kind of integrals have to converge and so, w of x is going to play the role of ensuring that such integrals converge. So, that is why this weight function is important.

And, so, let us see just based on some very general observations already several interesting properties can be brought out, right. So, let us suppose we consider just you know a finite number of elements in this set. So, the first n elements so, we look at C naught of x, C 1 of x, C 2 of x all the way upto C n minus 1 of x. So, the orthogonality condition already implies many you know constraints on this set right.

So, some interesting properties immediately follow. First of all we can immediately say that this set of polynomials is linearly independent, right. So, the argument is you know it is a; it is a proof by contradiction.

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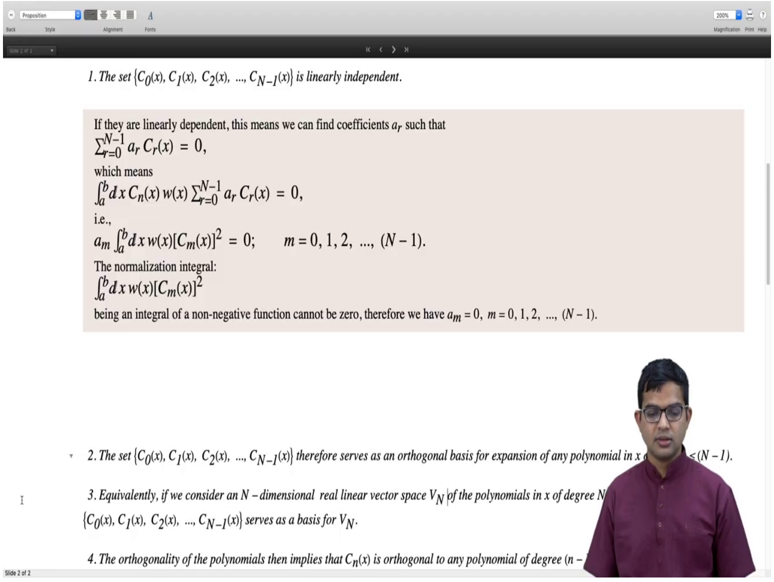

Suppose, the set is not linearly independent, so that means, they are linearly independent then we should be able to find some sort of coefficients a r such that summation over r going from 0 to n minus 1 a r C r of x equal to 0, right.

So, these coefficients are non-trivial in the sense that not all of them are 0, but if all of them are 0 then I mean that is not going to make it linearly dependent, right. So, that is the definition of linear dependence is that you will be able to find a set of nontrivial coefficients a r such that you know if you multiply these coefficients with these polynomials and add them up you should be able to take it to 0, right.

So, now we can bring in this weight function and some other generic polynomial C n of x from the left side and then integrate within this interval of relative which is relevant to this set of polynomials. So, from a to b dx C n of x w of x summation over this is 0 because you know the coordinate the sum itself is 0 and now because it is a finite sum you can change the you know order of integration.

So, the summation will come to the left of the integral and so, what it effectively means is this integral a m 0 a to b dx w of x C m C m of x the whole squared is equal to 0, right. So, where we have exploited this orthogonality property, right.

So, it is only when C n is equal to r thus you know do you get a you know potentially non-zero value. Well, in fact, we will argue that it is a non-zero value and every other one of them, if r is not equal to n then this condition immediately sends that object to 0 and so, there is only one such term which will survive.

And, so, this m can of course, take all these different values 0, 1, 2 all the way up to N minus 1 and so, now, the key point is that this is the only way for this product. So, there is some coefficient times this integral which is 0, but now we will argue that this integral cannot be 0, right.

The reason is that this is what is called a normalization integral, right. So, you have this normalization integral and the only way for this to be 0 is if this weight function itself were 0, but it is that is a trivial scenario, we do not want to consider the situation like that and if this weight function is you know is non-negative object and so, this is a you know a polynomial squared.

So, this polynomial squared will cross the x axis at most m times, right. So, there is no way that this quantity can be 0. So, that means, a m is necessarily 0 the only way this equation will hold is if a m is 0 and which in turn means basically it is a contradiction with our statement that you know we will be able to find some non-trivial coefficients a r such that this is equal to 0 that is not possible.

So, the only way you can ensure this type of a result is if all a r are 0 which is the statement that the set of polynomials is a linearly independent set, right. So, it is a linearly independent set and in fact, it forms an orthogonal basis for expansion of any polynomial in x of degree less than or equal to n minus 1, right. So, if you take this set of all these polynomials so designed C naught of x, C 1 of x so on all the way up to C N minus 1 of x, it will serve as an orthogonal basis for expansion of any polynomial in x of degree less than or equal to 1, right.

So, the idea is that basically you can think of a linear vector space which is N-dimensional a real linear vector space of polynomials in x of degree N minus 1 or less and the set this set C naught of x, C 1 of x, C 2 of x so on all the way up to C N minus 1 of x will serve as a basis for V N, right. So, that is the you know that is a consequence of this the orthogonality basically of all these polynomials.

Now, basically the orthogonality of these polynomials implies that you know if you can take any polynomial and you know expand it in terms of these polynomials you know these are the basis vectors of your space.

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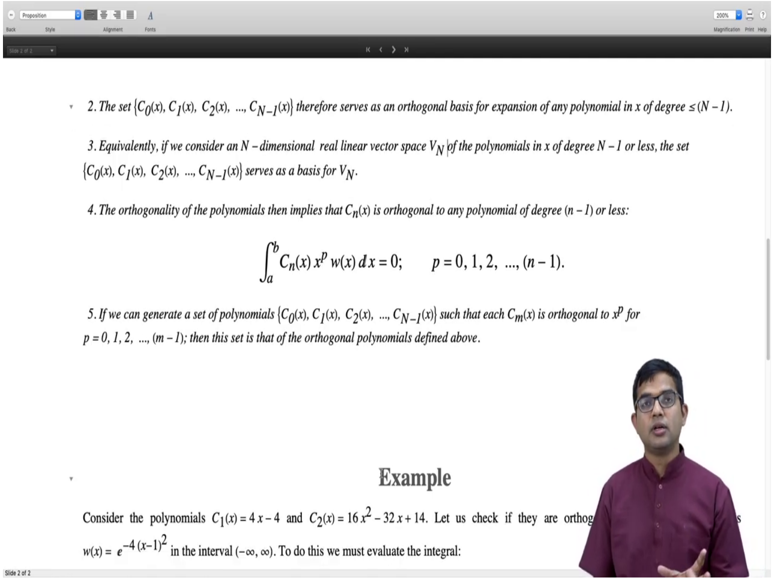

Then in fact, what you can do is you know write down you know this equivalent relation. So, what it means is, so, whenever you have a C n of x, where n is greater than you know any of these ps. So, consider x to the p, where p is an integer which is less than N. So, x to the p is going to be orthogonal to C n of x, right because after all C 1, C 2, C naught C 1 C 2 all the way up to C n minus 1 contain terms of this kind x to the p, right.

So, in fact, it is not just all these polynomials which are orthogonal to C n of x, but. In fact, every x to the p where p is less than or equal to n is going to be orthogonal to C n of x, right. So, the reason is that you can expand x to the p in terms of these polynomials which are all of lower order.

So, because you know all these polynomials up to p form a basis and therefore, x to the p is something which can be expanded in terms of these polynomials and therefore, when you multiply with C n of x and then you multiply by w of x and take this integral from a to b. Since x to the p itself is expressed in terms of all these polynomials each of them term by term will go to 0 and indeed in fact, you can argue that x to the p is orthogonal to C n of x whenever p is less than n, right.

So, this gives us a way to actually sort of generate these polynomials. So, what you have to do is take every C n of x to be you know orthogonal to every x to the p which is where p is less than n, right. So, if you can generate a set of polynomials C naught you start with C naught and you go to C 1 and make sure that C 1 is orthogonal to all x to the p, where p is less than 1. In this case that is just p equal to 0 if you go to C 2 of x you must ensure that C 2 of x is orthogonal to x to the 0 and x to the 1, alright.

So, if you can do this, then it is actually the same set that you are generating. You do not even have to go and work with C n being directly orthogonal to you know all C p, where p is less than n.

So, you can simply work with these powers x x to the 0, x to the 1, x x squared, x cube, and so on and then you are done, right. So, this is an immediate consequence of this vector space structure which comes about because of the orthogonality of this polynomial.

Let us look at an example to illustrate what this means is a rather simple example just to tell you that you know orthogonality is dependent on the weight function and on the interval that is being considered. Suppose you consider these two polynomials C 1 of x which is 4 x minus 4 and C 2 of x see some quadratic function 16x squared minus 32x plus 14 and suppose we consider this weight function e to the minus 4x minus 1 the whole squared and the interval is taken to be all the way from minus infinity to plus infinity.

So, if you want to check if these two functions are orthogonal to each other with respect to this weight function in this interval, what you have to do is carry out this integral.

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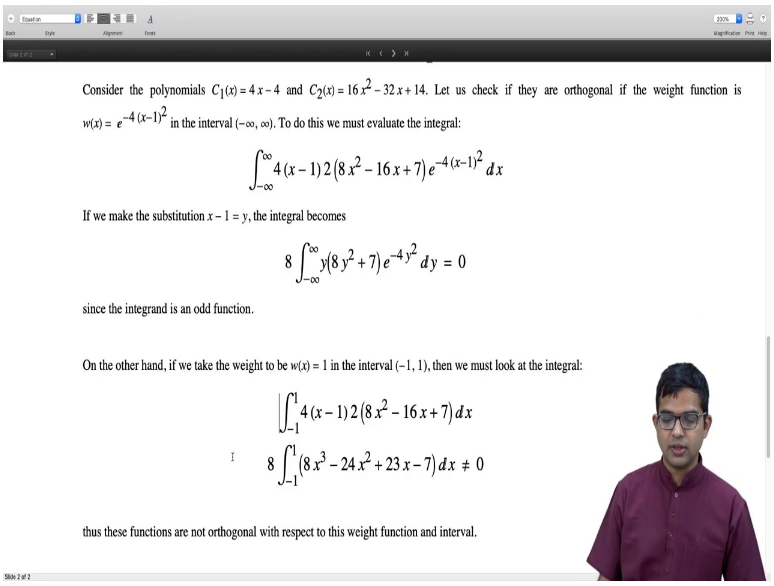

And, so, you will find that indeed you know if you make this substitution x minus 1 equal to y then the integral becomes so, 8 comes out then in place of x minus 1 you have a y, then you have an 8y squared plus 7 times e to the minus 4y squared dy. And, indeed we can actually immediately argue that this is 0 because integrand is an odd function, right. So, I mean there are other quadratic functions too which will give you this, right.

So, whenever any quadratic function which you can construct such that you know this integrand overall if it is; if it is going to be an odd function. So, which will happen if you know there is a y and there is a square of y if it comes in, then indeed this is going to be a 0. So, you will be able to find many quadratic functions which are orthogonal to this function with respect to this w in this interval, right.

So, but on the other hand if you consider the same two functions, but with a different weight in a different interval so, w of x equal to one and in interval minus 1 to 1, then we would have to look at this integral and with these limits. So, now, you see that the way to carry out this integral is to actually work out this product. So, what is a quadratic term times a linear term will become some cubic term.

So, then we see that although x cube and x are odd functions, but you still have these x squared and 7 sitting here which means that it is not going to be 0. You can work out the answer, but the answer is not important. So, we are just trying to argue that you know the same two functions are orthogonal to each other with respect to a certain weight function in a certain interval, but they are not orthogonal with respect to another weight function in a different interval, right.

So, it is important to specify ok.

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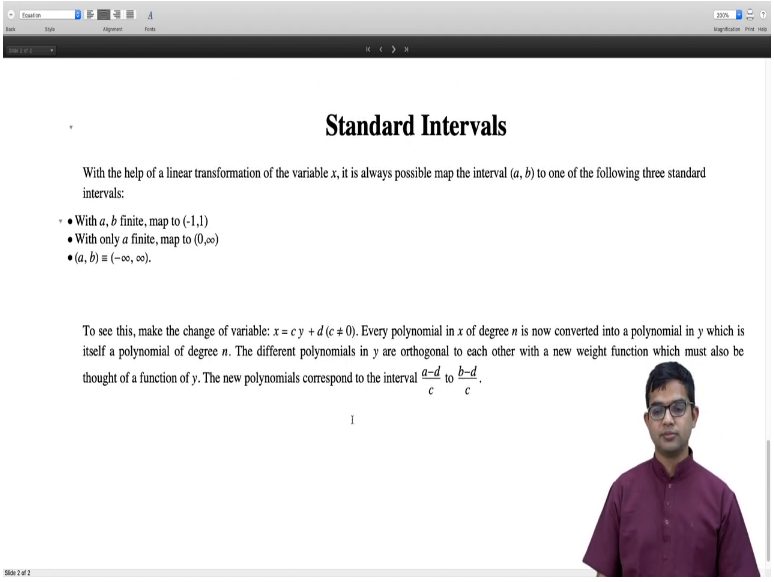

So, I mean we have started with a very sort of general interval in mind, but in fact, in print in practice it is useful to convert this interval to one of three different types and it is always possible to do this, right. One is if a and b are finite, it is useful to just simply map them to the interval minus 1 to 1, right.

So, there is a way to do this there is a linear transformation with which you can do it and if you have you know a is finite, but b is plus infinity then you just send a to 0, it is convenient to work in this manner and if both a and b are minus infinity and plus infinity that is ok. So, you leave it as it is, right.

So, the reason why this you know is possible is because you can make this change of variable you can write x to be cy plus d and c naught equal to 0. So, all these polynomials you know every polynomial in x will now become a polynomial in y and its degree is unchanged right after all it is just a linear transformation.

Now, the different polynomials in y are orthogonal to each other, but with a new weight function right which must also be adjusted and so, the idea behind doing this is to, you know, work with these new intervals right. So, if you know x were taking the value a y is going to take a minus d over c and when x takes the value of b y is going to take the value of b minus d over c.

And, so, you can convince yourself that you can always put this a minus d over C to be minus 1 and b minus d over c to be plus 1 provided both a and b are finite; if one of them is infinity then you know you can shift the other to 0, right. So, it is convenient to reduce this general nature of a and d and work with these three specific kinds, right. So, this is what we will do: build our structure of orthogonal polynomials using these three different types of intervals.

Ok, that is our short introduction to how we are going to start working in from very general principles, how we will try to you know look at orthogonal functions then we will see how to construct them and then look at some of their properties and look at specific sets of orthogonal polynomials as we go along. That is all for this lecture.

Thank you.