Mathematical Methods 2
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Complex Variables
Lecture - 38
Evaluation of Integrals

So, we have seen how the residue theorem allows us to compute contour integrals of apparently difficult looking functions, difficult looking contour integrals you know fairly simply by just computing residues at certain strategic points right.

So, whenever we have information about all the isolated singularities, then we basically are able to compute complicated looking contour integrals with ease. So, in this lecture, we will see how we can exploit this to obtain some interesting results for you know integrals involving just real variables ok.

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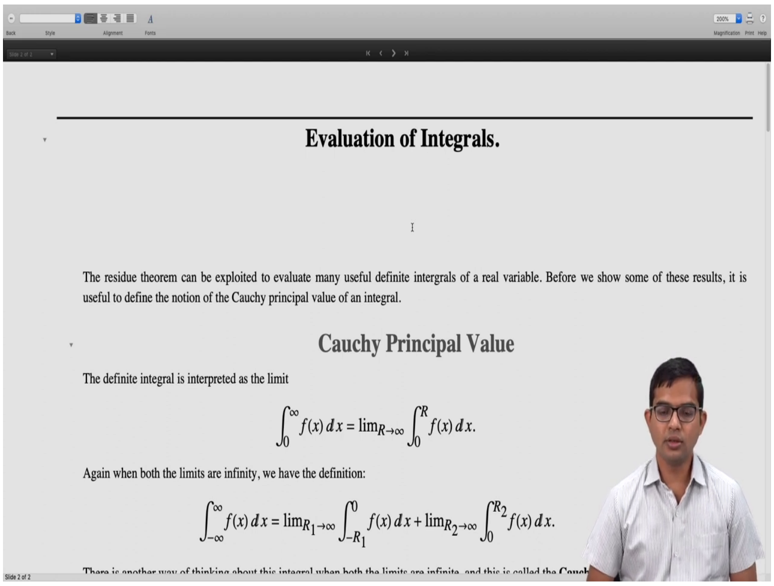

So, before we get into how to go about this, first there is a useful notion which is worth making precise right.

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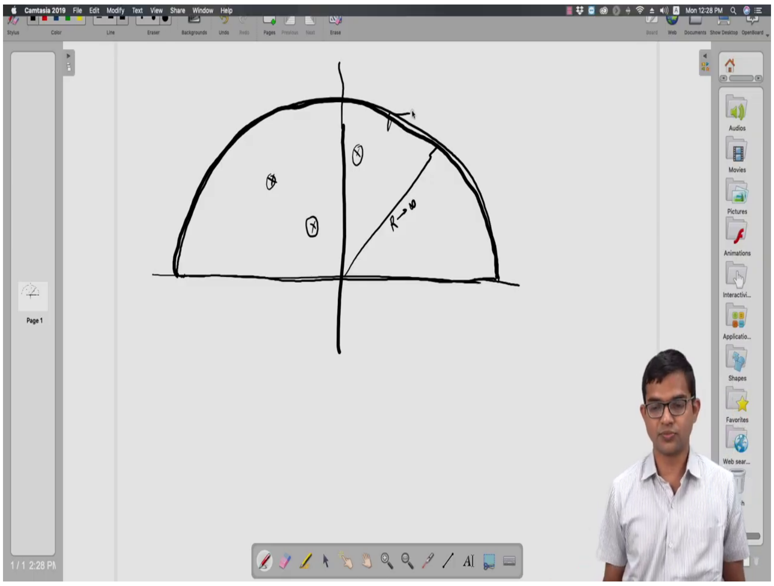

So, oftentimes, we are interested in working out integrals where the limits of integration go from minus infinity to plus infinity. So, what we want to do is be able to come up with a contour you know along the x axis which actually comes all the way from minus infinity goes all the way up to plus infinity, but not quite right, we have to come up with a limiting procedure.

So, we will consider a curve like this and say if by some means, we can argue that you know we have a some integral like f of x dx, then we elevate this f of x to f of z and I will make it a function of a complex variable and then, perform a contour integral around this and if by some means, we are able to argue that in the limit of this radius R becoming very large right.

So, in the limit of R tending to infinity, if we can somehow argue that the contribution which is coming from here is 0 and then basically, we will get the value of this integral along this x axis for free if we are able to work out the contour integral which in turn can be done by just finding out all the poles right.

If there are a bunch of singularities sitting at these points, we just need to evaluate the residues at these points and then, we have the answer right. So, that is basically the philosophy of this method. But before we do that, it is important to; it is important to discuss the idea of the Cauchy principal value right.

So, when we have an integral like this, some f of x dx from 0; going from 0 to infinity what we mean is we do the integral from 0 to R f of x dx and then, take the limit R going to infinity right. If we have both the limits plus and minus to be infinite, then what we mean by doing this integral minus infinity to plus infinity f of x dx is you know separately you break it down from minus infinity to 0.

And then, from 0 to infinity and then, ascribe meaning we need to take the limit R 1 tending to infinity and R 2 tending to infinity separately, you know the first integral is from minus R 1 to 0 and the second one is from 0 to R 2 right. So, this is what is meant by this integral.

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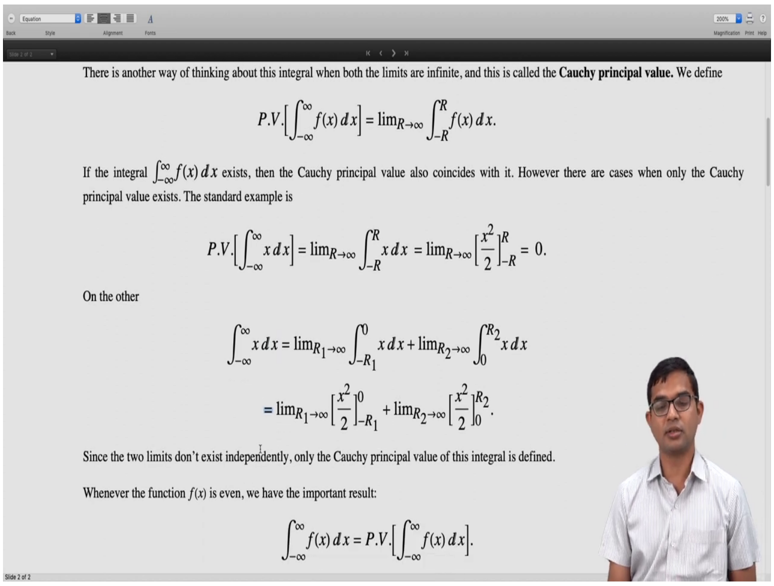

But often it is also useful to think of another quantity which is very closely connected and that is called the principal value of such an integral. So, that is called the Cauchy principal value and what the way to define this is you know just put one R so, you box this function f of x between minus R to plus R and take the limit R going to infinity that is just one of these; one of; one limit is taken right.

So, I mean it might look like really both are the same and often they are right, but there are cases where these two quantities are not quite the same right. So, if the; if this integral exists, if both of these limits exist, then for sure it is the same as the Cauchy principal value, but the other way around may not be true. So, let us look at an example right.

So, the standard example is to consider something like just x dx and you go from minus infinity to plus infinity. So, the principal value is simply you know R tends to infinity minus R to plus R x dx and, but this is x squared by 2 minus R 2 plus R and R tends to infinity, but x R squared minus R squared is basically 0 so, does not matter how large R is, this quantity 0, the principal value is 0.

But on the other hand, if you were to evaluate you know the other limit minus infinity to plus infinity as minus R 1 to 0 and R; 0 to R 2 separately and take these limits R 1 and R 2 to go to infinity, then you get a different answer you see x squared by 2 0 and minus R 1 and limit x squared by 2 from 0 to R 2 right.

So, you have a scenario which is you know it is like minus infinity and plus infinity right so, which is well, I mean it's so, it is 0 minus R 1 square so, it is going to be minus infinity. So, you cannot add these two and there is no; there is no well-defined answer for this. So, it is actually indeterminate right. So, these do not; these two limits do not separately exist and so, you say that this is not defined right, but the Cauchy principal value of this integral is defined right.

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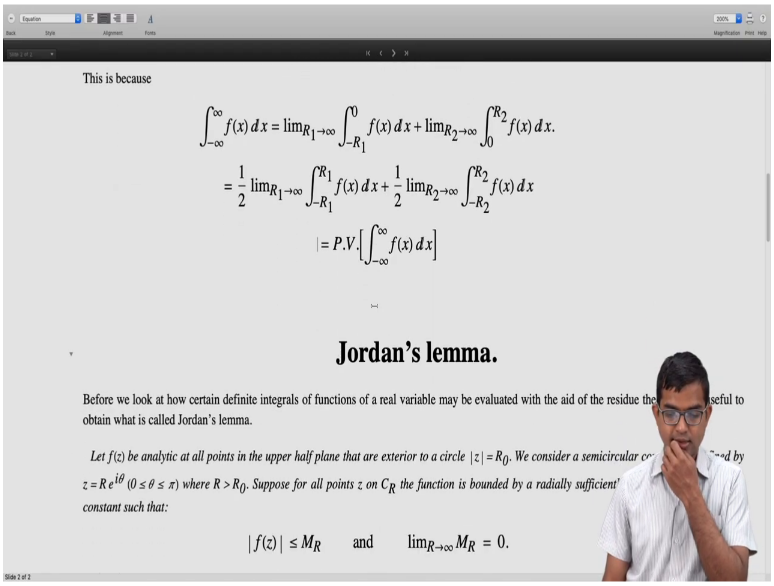

There are; there is one special case where for sure you know this integral is the same as the Cauchy integral principal value and that is when f of x is even right. So, the way to see that is if f of x is even I mean we can write minus infinity to plus infinity f of x dx as this according to the definition.

And then, because f of x is even so, we can write integral minus R 1 to 0 f of x dx is same as half of minus R 1 to plus R 1 f of x dx and again R 2, 0 to R 2 is the same as going from minus R 2 to plus R 2 and then, dividing by 2. Now, but each of these is separately yeah, you know this is half times the principal value and this is again half times the principal value.

In place of R 1 and R 2 here, we can just call it R and then, we immediately see that this is basically the principal value. So, half the principal value plus half the principal value, this the principal value. So, you get the principal value of minus infinity to plus infinity f of x dx right.

So, the reason why we want to clarify this is because you know when we are coming up with contour integrals, often it is really the principal value that we are working with right. So, we want to come up with a contour and then, take the limit R going to infinity right. So, that is what, that is the reason why we want to make this notion clear.

Now, I said the other ingredient in being able to evaluate such integrals along the x axis is to be able to convince ourselves that this integral along this you know this large semicircular region that must go to 0 right. So, that is where this thing called the Jordan’s lemma comes into play. So, I am going to try and motivate a somewhat more complicated example where Jordan's lemma is important.

But there are scenarios where even without the Jordan’s lemma, you can directly sort of argue that you know because R is becoming very large, the contribution along this semicircular part is going to go to 0 right so, there is a way to argue for that, but let us look at what is called the Jordan’s lemma right often, we are interested in integrals of this kind right.

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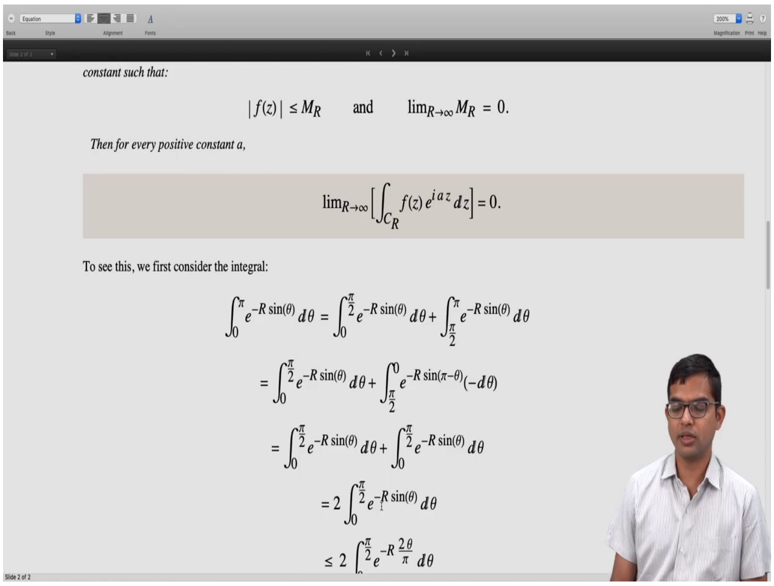

So, we are interested in integrals of this kind f of z times e to the i a z right. So, this you know comes a lot when we are working with Fourier transforms for example, right.

So, we are interested in f of x times e to the i a x right and in that context, we do encounter you know integrands of this kind f of z times i a z. So, now, if this function obeys certain conditions which we will state immediately, then we can argue that the value of this integral along this semicircular arc is going to be 0 right.

So, first of all, we consider this function to be analytic at all points in the upper half plane that are exterior to some circle. So, what it means is all the isolated singularities, all the poles which occur in the upper half plane are located in some finite region right, beyond that you will always be able to come up with some sufficiently large circle or semicircle beyond which there are no singularities, it is completely analytic right. So, that is the type of scenario you are looking at.

And then, we consider a semicircular contour C R whose radius R is greater than this R naught right. So, R that means, that R contour basically for sure includes all the poles, all these isolated singularities of this function f of z are contained inside this and then, R function R C R you know the function on C R is bounded by a radially sufficiently rapidly decaying positive constant.

So, what it means is mod of f of z is less than or equal to M R right. Now, this is mod this bound is going to depend on R, but that bound is such that it becomes smaller and smaller as R becomes larger and larger and in fact, and in the limit of R going to infinity, M R is actually 0 right.

So, basically this function is you know it is not some weird function which will blow up at a infinity or something, it is going to keep on decaying basically that is what it means and if that holds, then what Jordan's lemma says is you know this then you can basically ignore the contribution coming from this semicircular large arc right.

So, let us see how this Jordan's lemma comes about, it ultimately connects to something called the Jordan's inequality right so, that is just a simple integral. So, suppose you are considering this integral 0 to pi e to the minus i; minus R sin theta d theta, you can write it as 0 to pi by 2 and then, pi by 2 by the same integrand and then, you can do a shift in theta. So, instead of you know writing it as this so, you introduce this change of variable for the second of these integrals.

So, in place of theta, you call it pi minus alpha and then, the limits will be pi by 2 to 0 and you might as well call it pi minus theta. So, in place of d theta minus d theta and then, you see that sin of pi minus theta is the same as sin theta and this minus sign goes here and basically, you get that this integral is the same as this. So, in fact, this integral on the left-hand side is 2 times this integral from 0 to pi by 2, same quantity e to the minus R sin theta d theta.

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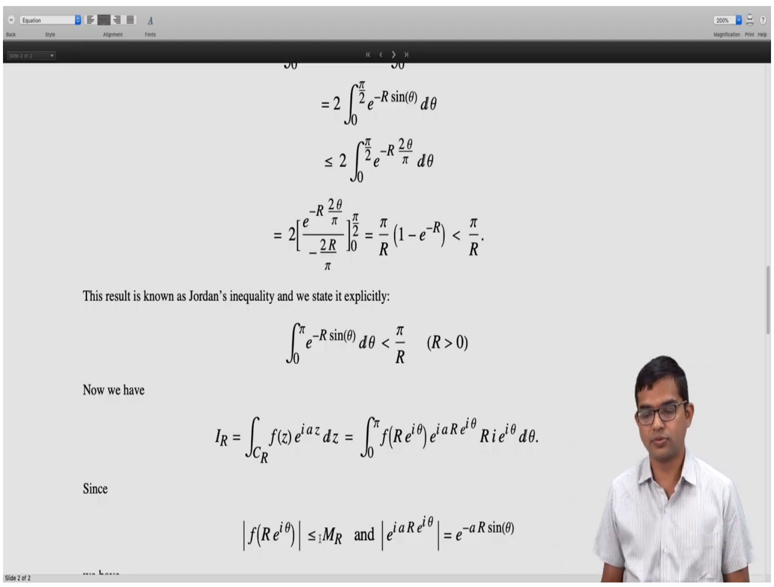

Now, you argue that in this region 0 to pi by 2, sin theta is always greater than or equal 2 theta by prime right. So, basically, one way to think about this is you know theta is 0 at 0 and 2 theta pi is also 0 at 0 and 2 theta pi is 1 at pi by 2 and sin theta is also 1 at pi by 2. So, basically, this curve is 2 theta by pi is like a straight line that you join between 0 and the point, the maximum point of the sine curve at pi by 2.

Now, sin is always going to be above the straight line between these two points as you can check by just plotting it for example. So, therefore, since sin theta is greater than or equal to theta pi in this entire regime so, this integral is going to be less than or equal to 2 times 0 to pi by 2 e to the minus R 2 theta by pi d theta, but this integral is something you can work out explicitly and we immediately have the result that this is you know this integral is equal to pi by R times 1 minus e to the minus R.

So, basically, we managed to immediately show that you know this is the Jordan's inequality, we have managed to show that whenever R is positive which is the case, here it is a radius and so, it has a positive value 0 to pi e to the minus R sin theta d theta is less than pi by R right. So, that is Jordan's inequality.

And our Jordan's lemma actually follows from this. So, what you do is you are you write down this contour integral I R over C R, C R is this big semi-circle f of z e to the i a z dz and then, you write it as 0 to pi d theta so, R i e to the i theta d theta in place of dz and then, we have in place of e to the i a z, you write it as e to the i a R e to the i theta and f of R e to the i theta also you have here.

So, but mod of f is given that this function f of z is going to be less than or equal to M R on this semi-circular region. So, f of z is decaying as you make R and R larger and larger, its magnitude is falling and on the other hand, magnitude of this quantity is basically just given. So, you have in place of e to the i theta, I can write it as cos theta plus i sin theta, cos theta will only contribute to the phase plus i sin theta will become e to the minus a R sin theta right.

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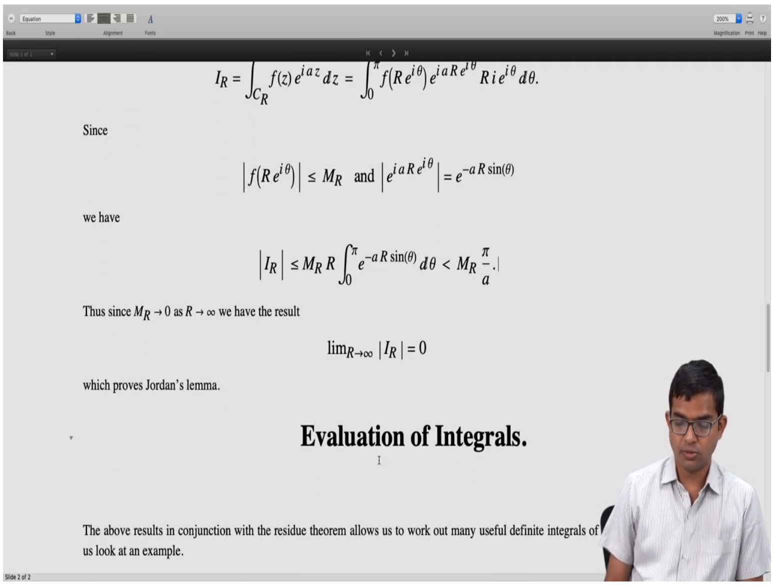

So, basically what we have is if I take the modulus of this, modulus of this integral is you know it gives a modulus of this guy or modulus of this, then we basically argue that modulus of this integral must be less than or equal to you know pulling out the moduli of the integrand outside right. So, this is somewhat like the triangular inequality generalized to integrals.

So, you can pull out this M R times R also comes out and then anyway so, this quantity is minus e to the minus a R sin theta d theta and this is something so, we have already seen that is the Jordan inequality will just give us pi by a R and R will cancel with this. So, we are left with M R times pi by a right.

So, now, comes the key you know argument from the other requirement. M R is something which falls off with R right. So, it is not enough that there is an R coming from this integral because it cancels with this R, but we have chosen our M R such that it is going to fall off with R.

Therefore, mod of I R in the limit R going to infinity is actually 0. Therefore, basically I R itself is 0. So, we can argue that you know the contribution from this big semicircular arc is going to go to 0 whenever this condition holds and often this condition does exist for many useful integrals.

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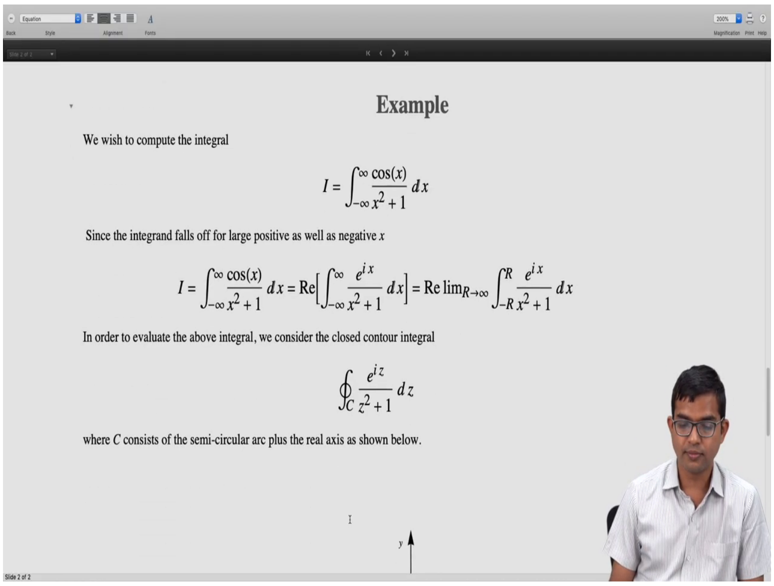

So, let us look at an example of how this plays out. So, we look at an example where the Jordan's lemma you know plays out. It is essential to make the argument complete. But there are other examples where a similar trick will hold and it is easier to argue for the vanishing value of this contour integral along this semicircular path even without invoking Jordan's lemma.

So, we wish to compute this integral let us say cos x by x squared plus 1 dx minus infinity to plus infinity, this is not an even function right, but then, it falls off at largely, it is not like f of x equal to x, it falls off on both sides. So, basically it is actually equal to the principal value right. So in fact, integral minus infinity to infinity cos x by x squared plus 1 is the same as limit R tending to infinity minus R to R cos x by x squared plus 1.

But here in this case, it is useful to think of this as the real part of e to the i x, cos x is the same as real part of e to the i x and then, you pull out the real part and then, you do this limiting operation R tends to infinity minus R to plus R e to the i x divided by x squared plus 1, then to the function that we must consider the function of a complex variable that we must consider is e to the i z divided by z squared plus 1 so, we wish to evaluate this contour integral with the contour integral e to the i z divided by z squared plus 1 dz.

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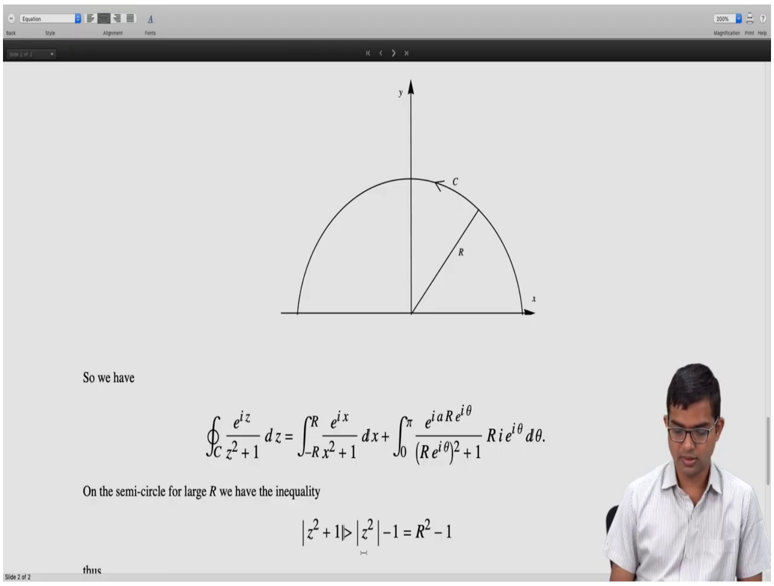

And what is the contour? The contour is this big you know big line segment along the x axis followed by the semi-circular lag and arc and then, you turn around completely and complete the loop right. So, and then we of course, want to imagine taking the limit R going to infinity in this case. Now, we have what is the contour integral along this entire path is actually nothing, but minus R to plus R so, that is the; that is the integral that we care about right so that we can take the real part of this.

So, this is the part which we care about plus 0 to pi, you know this whole stuff, R i e to the i theta d theta, but then, we see that you know mod of z squared plus 1; mod of z squared plus 1 so, that is what is in the denominator, mod of z in the denominator here mod of z squared plus 1 is greater than mod of z squared minus 1 so, that is the triangular inequality right. So, we have put a greater than symbol here and, but mod of z squared is the same as R squared on the semicircular region.

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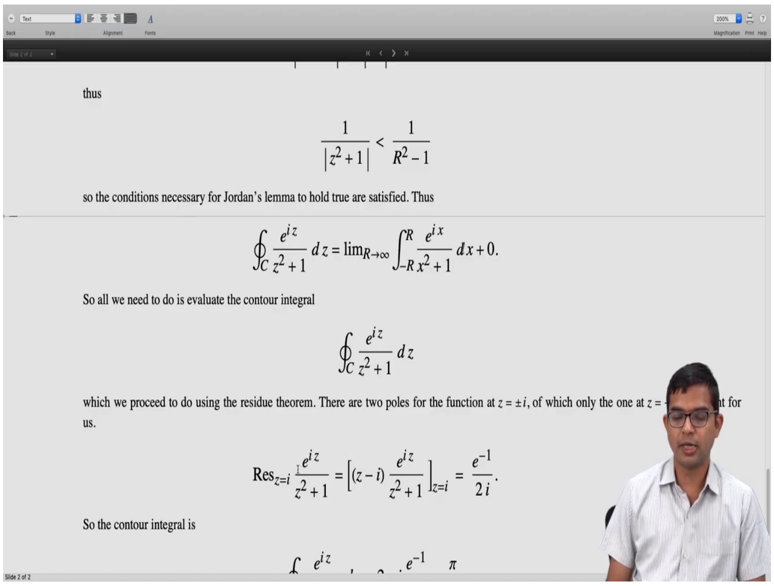

So, therefore, we have this result that 1 over mod of z squared plus 1 is going to be less than 1 over R squared minus 1. So, basically, we want to argue that this function 1 over z squared plus 1 is falling off rapidly enough so that the condition which is given in Jordan's lemma holds and indeed this is falling off rapidly as R becomes larger, integrand is going to become smaller and smaller right. So, we will just treat e to the i z as separate.

And so, we immediately see that this contour integral you know the term the contribution from this semicircular arc, this big semicircular arc is actually 0 because Jordan's lemma holds we invoke Jordan's lemma and then, we have the result that this contour integral over this entire closed region is actually nothing, but equal to the integral that we are interested in.

So, all we have to do is work out this contour integral, but we know how to work out this contour integral because we can use the residue theorem. So, the residue theorem tells us to find all the poles. There are two poles for this function, one of them is at plus i and the other is at minus i and the one at minus i is not within your region of interest, it is only plus i that counts.

So, by the way this same kind of a calculation could have been done by completing the loop in the other direction as well. So, there are certain problems where it's more useful to go in the other direction. So, for as far as this problem is considered, it does not matter which direction you go, Jordan's lemma would hold even in the other direction as well provided you can argue for the you know some conditions which are very similar, but you know with the direction being different.

So, here, the residue is simple because all we have to do is multiply by z minus i right, we are considering the pole at z equal to i so, multiply by z minus i and put the value z equal to i.

So, you immediately get this to be you know the denominator will become z plus I so, when you put z equal to I, it becomes 2i, numerator is e to the minus 1 residues is there. So, the value of the integral very simply is just 2 pi i times e inverse divided by 2i which works out to just pi by e, very straightforward.

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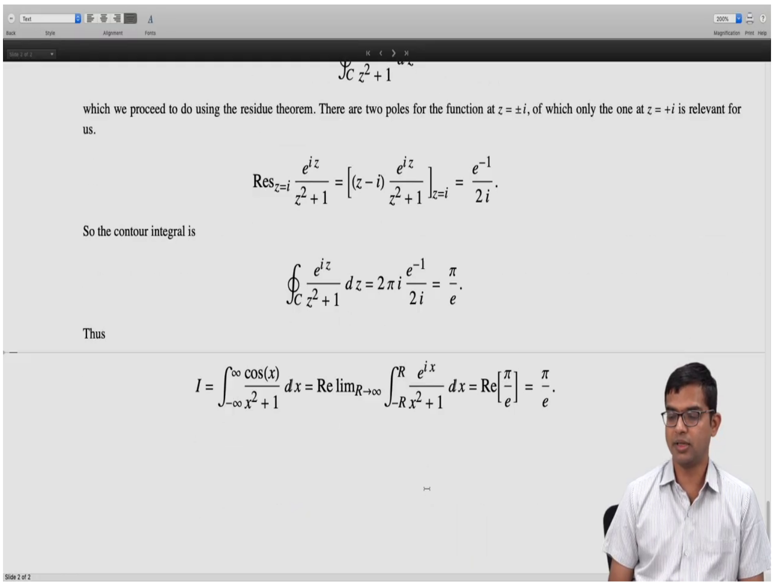

So, what we have managed to show is this complicated looking integral minus infinity to plus infinity cos of x divided by x squared plus 1 is actually we have managed to show like the Fourier transform in some sense of this function 1 over x squared plus 1 is well, I mean you have to be careful because you know Fourier transform also involves some k or some other coefficient right which can also be written down.

But in this case, the specific integral which we started with is this integral which is obtained as the real part of this quantity which we already worked out and that is just the real part of pi by e is just pi by e. So, the final answer is very straightforward and in fact, the technique involved is also very straightforward.

It is just that we have to make use of these results along the way, including Jordan’s lemma. Also one has to be careful because one is considering the idea of a principal value, Cauchy principal value.

Once we have this, actually the machinery allows us to calculate these quantities with great ease once we have all the tools right. So, that is all for this lecture. We looked at how we can apply the residue theorem, how certain complicated looking integrals of real variables can also be worked out.

Thank you.