Mathematical Methods 2
Prof. Auditya Sharma
Department of Physics
Indian Institute of Science Education and Research, Bhopal

Complex Variables
Lecture - 37
Residue theorem

Ok, so we have been leading up to the Residue theorem right in the form of our discussions relating to Taylor series, Laurent series, expansions of functions which are analytic and which have some analyticity, non-analyticity in some region, but where analyticity is available in a in an annular region when we have seen how a Laurent expansion becomes possible.

So, in this lecture, we will see how when we have functions which have isolated singularities right. So, it becomes possible to work out certain contour integrals you know using just the residues right. So, we have looked at the idea of a residue, but we will see how it plays out, why it is so important is something we will discuss in this lecture ok.

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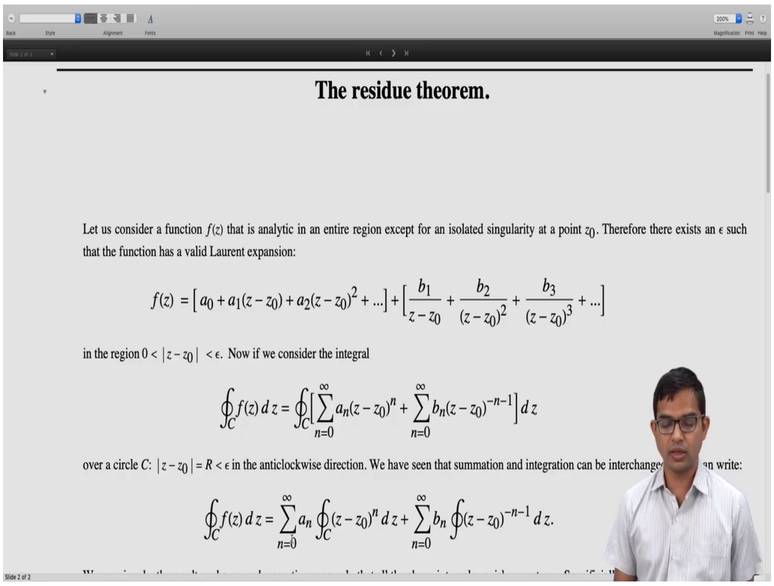

So, if you have a function f of z that is analytic in an entire region excepted for except for an isolated singularity at some point z naught, then we know that there is a valid Laurent expansion which we can write down where you know all these positive powers and z minus z naught you know they form the regular part. And then you also have these coefficients b, b 1, b 2, b 3 and so on. And they together form the irregular part right, so that is where the singularity is.

So, b 1 over z minus z naught b 2 over z minus z naught square so the whole squared plus and so on right. And this expansion is valid in some region 0 less than mod of z minus z naught less than epsilon because you have an isolated singularity at z naught.

So, now, we will see how among all these coefficients, b 1 has a special name associated with it because it has some special importance as we will see. And so the way to see that is to actually just take a contour integral you know over a circle let us say right.

So, over a circle whose radius is r which is centered about this point z naught and where this radius R is less than epsilon right, so that you are in this region where this expansion has its validity. And now we have seen that you know you can go ahead and exchange the integral and summation because of these uniform convergence properties.

And so we are performing this contour integral in the anticlockwise direction right. So, therefore, we can go ahead and write this contour integral of f of z dz over z to be just summation over n, n goes from 0 to infinity a n contour integral z minus z z naught the whole power n dz plus summation over n again n goes from 0 to infinity b n contour integral of z minus z naught the whole power minus n minus 1 dz ok.

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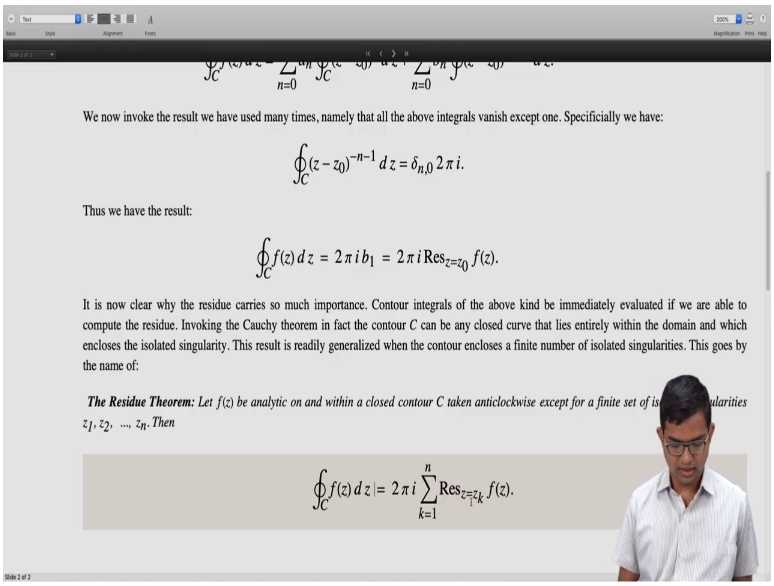

So, now we invoke a result we have used many times. So, we have seen that whenever you have a function like z to the n dz and if you take a contour integral around the origin which in a closed contour which encloses the origin, we have seen that for all integer values or n you know this contour integral will vanish except when n is equal to minus 1, 1 over z is special and that gives you 2 pi i.

And so this result generalized to some other point z naught. It is just the result here which is that the contour integral over C z minus z naught the whole power minus n minus 1 dz is equal to delta n comma 0 2 pi i right. So, here I am not even specifically stating that all these positive powers go to 0 right. So, it is self-evident because after all these functions z minus z naught to the n you know they have this analyticity properties right.

So, in any case, so this, this result holds. And therefore, we have this contour integral f of z dz is just given by 2 pi i times b 1 right. So, in this case I mean we have a single we have a singularity sitting at z naught.

So, it is I mean even with the singularity present, it is still true that even when you take a contour integral of C over C z minus z naught the whole power n dz is going to be 0 for all these powers except for 1 over z minus z naught right, so that is why this coefficient b 1 is of such great importance right.

So, I mean this result is something that you know we have worked out explicitly by you know writing down the contour integral as just an integral over theta. So, right in other words write down z as z naught plus r times e to the i i theta and then perform the integral and then explicitly verify that indeed all of these integrals will just vanish except 1 over z minus z naught when you get 2 pi i right.

So, we have this result that the integral over C f of z dz is 2 pi i b 1, but b 1 has a special name and it is called the residue of this function f of z at this isolated singularity z equal z naught. So, this is the reason why you know this coefficient has a special name.

So, in fact, this contour does not have to be a circle. It can be an arbitrary shaped object as long as it is a simple closed contour which encloses this singularity and it encloses no other singularity right. It is, it should enclose only one singularity, and it should also not, you know, not pass over a singularity. It must lie entirely within the region of analyticity right and enclose this singularity inside it.

So, in fact, this result can be generalized to include many singularities, and so that is the theorem which we state it is the residue theorem. So, if f of z is analytic on and within a closed contour C taken anticlockwise except for a finite number of isolated singularities. So, basically it is a nice function, it is a, it is an analytic function except at these points right.

So, what the residue theorem tells us is if you are taking a contour integral of such a function over you know which encloses a finite number of these isolated singularities you just simply take, so it is so it is actually summation over this. So, the result is equal to I should have said summation over all the b i s, so which adds up to which is going to give you 2 pi i times summation over k residue at each of these singularities z equal to z k f z right.

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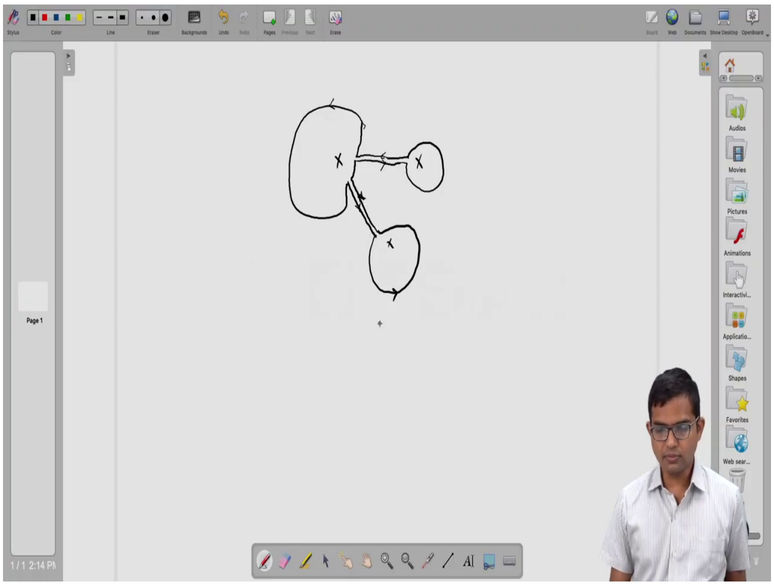

So, basically what we are saying is, you have a function which is analytic in this entire region except that, let us say there is a singularity here, there is a singularity here, maybe there is a singularity here right. So, yeah you can come up with you know there is this theorem which we discussed which allows us to deform these contours. So, whatever the value of the integral is around this contour is basically the same as the value of, so I mean let me try to redraw this.

Let me actually erase this entire thing, erase this entire thing. And then I will argue that you know the value of this contour integral is the same as if I were to do something like this go around here. And then come back down this path, and then go around here, and come back around here, and then go like this. And then maybe you know, come up with a contour like this.

And then basically you can make each of these you know small circles you can make it exactly like a circle and very tiny ones which pertain only to those isolated singularities in order to take care of the direction, but basically you can argue that the value of the contour integral along paths like these will cancel each other out right along this direction and along this direction, opposite direction. So, here it's coming in this direction. So, here it is going to be coming in this direction. So, they cancel each other out.

So, in the end you are left with just the contour integral around this point, around this point and around this point. All of them have to be added and then each of these separately by the residue theorem is just going to be this 2 pi i times the residue at that point.

So, therefore, it is clear that you know the overall result is going to be 2 pi i times the sum of the residues because they are finite in number. Let us look at a few examples where this theorem holds out.

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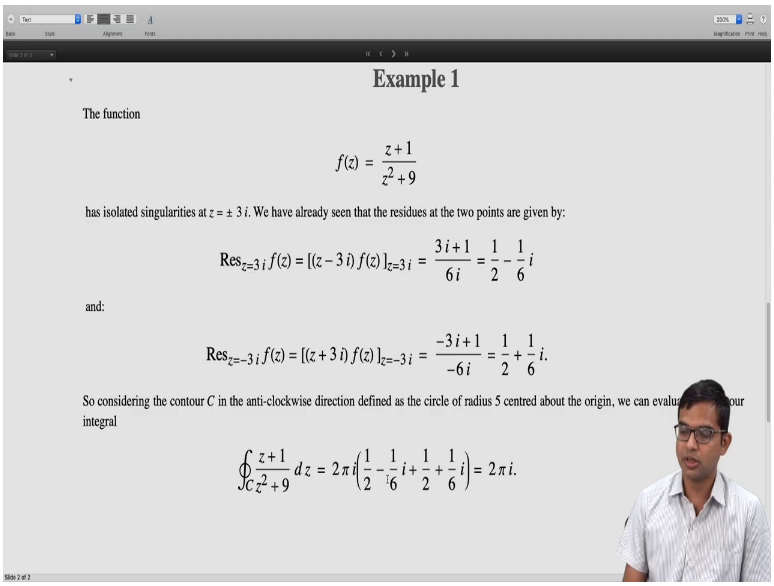

So, suppose we consider a function like f of z is equal to z plus 1 divided by z squared plus 9. So, we have already evaluated the residues. There are two singularities, there are two isolated singularities for this function one of them is located at plus 3 i, and the other one is at minus 3 i. Both of them are simple poles.

So, the way to evaluate the residue at these two points is to simply multiply by you know in the first case by z minus 3 i, and take the value of this new function at the point z equal to 3 i. So, we have already done this exercise, and the residue we found was equal to half minus 1 by 6 i.

Again to find the residue at the other point z equal to minus 3 i, it is going to be z plus 3 i times f of z, and then we have to put the value z equal to minus 3 i. So, which we evaluate to be half plus 1 by 6th i. So, considering the contour C in the anticlockwise direction, it has to be a contour which encloses both of these singularities.

So, let us take for simplicity a circle of radius 5 which is centered about the origin. So, clearly both of these singularities which are at plus 3 i and minus 3 i are also included. And so we can evaluate this contour integral to be just 2 pi i times half minus 1 by 6 i plus half plus 1 by 6 i. Both minus 1 by 6 and plus 1 by 6 i is canceled. So, we are just left with 2 pi i right.

Very simple answer for what looks like an apparently quite a difficult problem. If you had to work it out the hard way writing down you know for z putting down you know r time z to the i theta i and so on, it should be quite a messy task, but so we have already reduced this sort of difficult problem to just calculating residues, so that is where the power of this approach comes from.

And also we know from this principle of deformability right. You did not have to be a circle, it could have been some other more complicated object if you have to evaluate that then it is a nightmare if you have to do that directly from first principles, but residue theorem allows us to compute it with ease.

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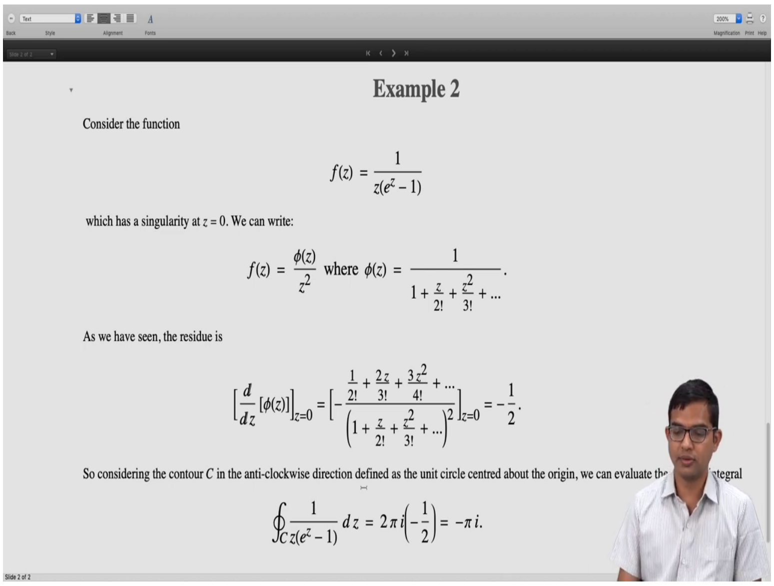

So, let us look at one more example. Again so this function has a pole of order 2 at z equal to 0 right. We have seen this 1 over z times e to the z minus 1. So, we can write this as phi over z divided by z squared, phi of z has this expansion 1 over 1 plus z 2 z over 2 factorial plus z squared over 3 factorial and so on.

As we have seen, the residue can be computed by taking the derivative of this quantity at z equal to 0, and it turns out to be minus a half right. So, you can go over this calculation of the residue again if you wish. So, you see that you know it is important to be able to calculate residues.

If you can work out residues, then you can work out these contour integrals with ease because of the residue theorem. So, since the residue we already have worked it out is minus half for this. So, if we consider the contour C in anticlockwise direction a simple closed curve which for our for simplicity we can just take it to be the unit circle centered about the origin in this case because there is only one singularity isolated singularity, and that is at z equal to 0, it is a pole of order 2.

We have already worked this out. And so it is simply a matter of writing down the answer now. If you were to take this contour integral, it is going to be 2 pi i times minus a half. So, it is just minus pi, very straightforward ok. So, there are more examples which we can consider, perhaps that is going to be part of the homework, but the key idea already comes out with these few examples right.

So, the residue theorem can be used to compute contour integrals of this kind which in turn you know used cleverly can be used to evaluate many interesting integrals of a definite integrals of a real variable right. So, some of those examples also we will look at.

But, as far as this lecture is concerned, it is primarily about seeing how the residue theorem plays out right. We have stated the residue theorem, we also argued for how it comes about, and showed a few examples of how to make use of the residue theorem.

Thank you.