Mathematical Methods 2
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Complex Variables
Lecture - 35
Isolated singularities

So, we have seen how when a function is analytic, there is a Taylor series of expansion available. And even, if a function is non-analytic about a point, you know we often have a Laurent series available for it. So, in this lecture, we will look closely at the idea of singularities which are isolated, right.

So, there are different types of isolated singularities possible. And so, this will come in handy when we discuss the next topic namely the residue theorem, right. So, in that context, it is useful to classify singularities and understand what kinds of singularities appear. And so, that is the topic for this lecture, ok.

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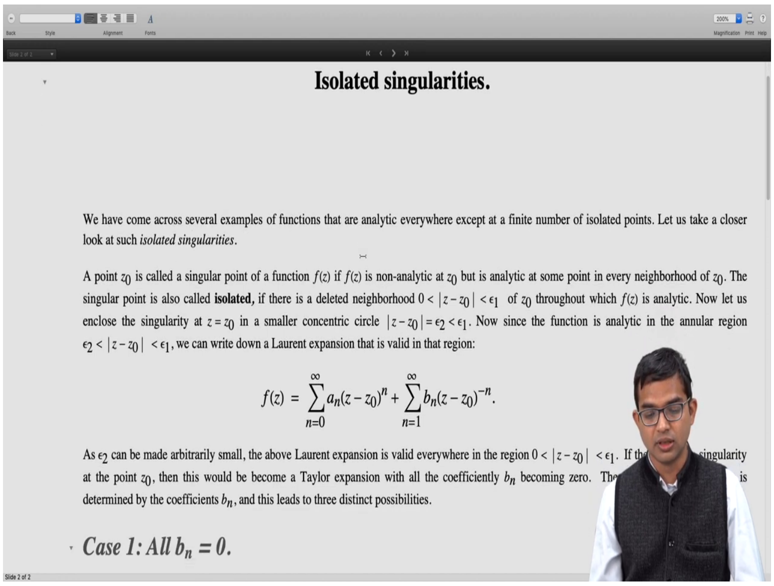

So, we have encountered lots of functions which have non-analyticities, but restricted to a finite number of points, right. So, a simple example is 1 over z, right. So, if you have a function like 1 over z, then it is well-behaved, it is analytic everywhere except at the origin, right. So, similarly there are other functions which we have seen where such non-analyticities are restricted to a finite number of isolated points, right.

So, let us define isolated singularities, a an isolated singularity a little more precisely. So, we call a point z naught you know a singular point if it is non-analytic, but its and its analytic at some point in every neighborhood around z naught, right. So, this is about a singular point. So, you will always be able to find some analyticity in the neighborhood, then it is a singular point.

But, it is also called an isolated singular point if it is analytic everywhere in its neighborhood, right. So, if there is a deleted neighborhood, it was a technical term. So, deleted neighborhood simply means some region 0 less than mod z minus z naught less than epsilon 1, you are able to find some radius and a circle of radius epsilon 1.

And every point inside this region is analytic except this point z naught. So, that is what is meant by this deleted neighborhood. Throughout which f of z is analytic, then you say it is isolated, right. But in general a singular point need not be isolated, right.

So, for example, if you have something like log z, right, so the function log z is it has a singularity at the origin, but it is also along an axis. So, it is not even defined along the negative x axis, if you take that to be your branch cut, right. So, you could of course, by convention choose a different branch or different branch cut. But in any neighborhood you will find other singularities as well, right.

So, that is why in general you can have singularities which are not isolated, but it would be isolated if you have you know that is the only singularity. I mean you may have another singularity sitting at a finite distance away, but all points in between these two singularities have to be analytic, right. So, this whole region around this point if it is analytic except that point then it is called an isolated singularity.

Now, since this function is analytic in this annular region, epsilon 2 you can always come up with some epsilon 2 less than mod z minus z naught less than epsilon, we can epsilon 1 we can write down a Laurent expansion and that is valid in this region. So, you have some expansion which has all these coefficients a n's and b n's.

You write down summation over n a n z minus z naught the whole power n, positive powers and then summation over b n; you know these are also positive you know in indices, but basically the powers are in the denominator. So, you can think of this as b and divided by z minus z naught, the whole power n and n goes from 1 to infinity, so in general, right.

So, since epsilon 2 can be made arbitrarily small, basically you say that this Laurent expansion is valid everywhere in the region, 0 less than mod z minus z naught less than epsilon 1.

So, this radius epsilon 1 is often you know as large as the distance to the nearest you know singularity from a, given singularity there is another point of non-analyticities somewhere else which is so, you just consider the nearest of these singularities and that would be this epsilon 1, but it could be a smaller one.

So, it does not matter how large or small this epsilon 1 is. The key point is that there is a whole circular region which encloses this point and in that entire region it is analytic and therefore, you are able to write down the Laurent expansion of this kind. So, now, if you are able to write down a Laurent expansion of this kind. So, the nature of the singularity is determined by these coefficients b n basically.

So, a n's that is the regular part which contains you know information about the regular part is contained in these coefficients a n, but b n carry information about the nature of this singularity and depending upon what kind of b n, the set of coefficients b n you have there are 3 different cases and its useful to look at these 3 cases in some detail. The first of these cases is something you already encountered which is when all these b n are 0, right.

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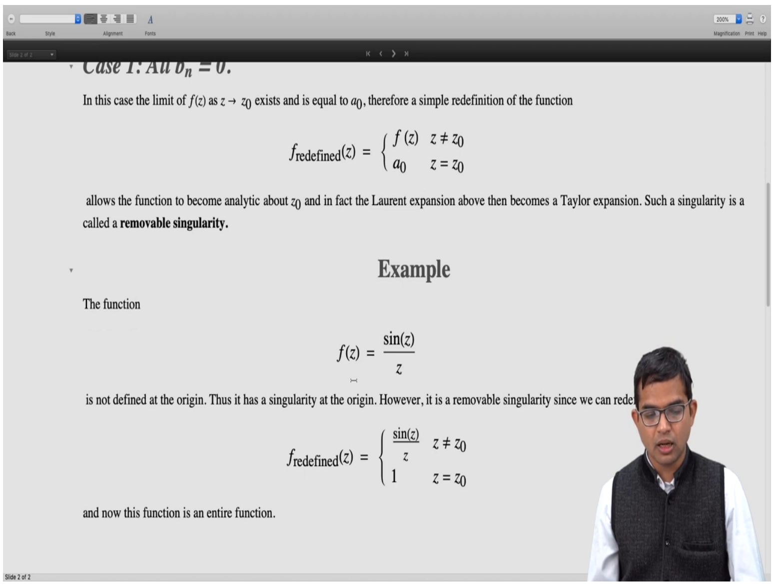

So, then you basically have a scenario where only the a n exists. And so, it is basic, it is like a Taylor series expansion except that there is a small certainty involved here and that is that the function may not be defined at this point. So, that you still have a singularity, but you know there is a it is what is called a removable singularity.

So, the reason is that the limit of this function f of z has z tends to z naught exists because it is going to be a naught. And therefore, if the function is not already defined to be a naught at that point, you just simply say that we call this function we equate this function to f of z as long as z is not equal to z naught, but we will just redefine it to be a naught at the point z equals z naught.

And if you do this then you get a well-defined Taylor series expansion, I mean the Taylor series expansion is already there and this function is an analytic function, it's analytic everywhere. So, this is why it is called a removable singularity. This is sort of a trivial kind in some sense, right. We already encountered this kind of a removable singularity. A simple example is that of the function sin z over z, right.

It is not defined at the origin because of the 0 by 0 form. So, it has a singularity at the origin, but you can just simply redefine this function to take the value of the limit at that point which you can easily extract, by simply writing down the Taylor series expansion for sin of z.

And then you see that the leading order is z which will go which will cancel with the denominator. And so, basically the limit of sin of z divided by z is going to be 1, as z tends to 0, and this is the same regardless of which direction you are approaching from. So, it is continuous and in fact, this is analytic everywhere, right. So, you have a well-defined Taylor series at all points. And so, this is an entire function.

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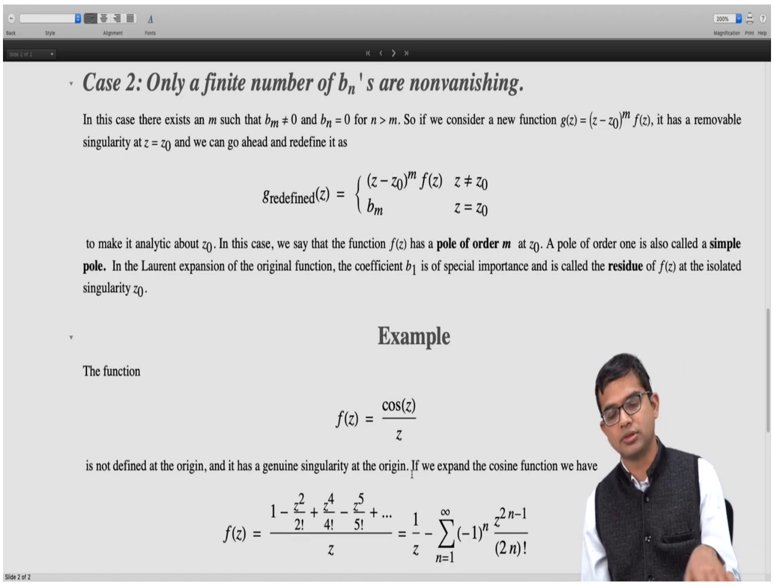

Now, let us look at case 2, when you have some of these coefficients the b n's are non-vanishing, but not all of them. So, a finite number of b n's are non-vanishing. Then, we will basically be able to find some largest of these coefficients, largest index coefficient. So, there exists an m, such that b m is not equal to 0, right.

So, I mean you do not have to have b m nonzero not equal to 0 all the way up to m, right, you find some particular m, whose b m is not equal to 0. But you must have the condition that b n is equal to 0 for all n greater than m, right. All higher order coefficients for sure are 0, lower or order coefficients some of them may exist, some of them may not exist.

Then, you have a scenario where you can simply multiply such a function at this point by this this power z minus z naught the whole power m, and then you are guaranteed that this new function g of z is going to have a removable singularity at this point z equal to z naught, right.

So, and then it becomes possible to redefine this function z, you know as z minus z naught the whole power m times f of z, if z is not equal to z naught and it is equal to b m, if z equal to z naught, right. So, b m is the coefficient which is critical in this for as far as this new function is concerned, right.

It is not a naught, but b m, right. So, b m will play the same role as a naught played in the previous discussion. And then it becomes analytic at z naught therefore, analytic in some region around it in the neighborhood. And so, in this case, we say that the function f of z has a pole of order m at z naught.

A pole of order 1 is called a simple pole, right. We have encountered simple poles. So, in the Laurent expansion of the original function; so, this is a very important definition if you wish at this point is that of the residue, right. So, I mean among all these coefficients when you have a Laurent expansion of this kind about an isolated singularity, all these coefficients have been important to some extent.

But there is one coefficient which is of great importance and that is this coefficient b 1. And this coefficient b 1 will have the, you know corresponds to the term 1 over z minus z naught, right. So, b 1 divided by z minus z naught is of paramount importance.

And so, this coefficient has a name and it is called the residue, right. So, you have, so the residue of a function f of z is simply defined as the coefficient b 1 in the Laurent expansion of this function when it is expanded about an isolated singularity.

Now, this is true in general, right. It does not have to be a pole. It is an isolated singularity. But when you have a pole of order m there is a way to compute this residue with some ease. So, let us look at an example. So, if you have this function cos z of z instead of sin z of z I can think of cos z of z.

Now, it is a real singularity. It is not of a trivial kind. It is a genuine singularity, and it does not have you know it is not defined at the origin, and there is no simple way to redefine this function at that point and get rid of the singularity it exists.

So, if you expand the cosine function, we have you know the series expansion, we already wrote down 1 minus z squared over 2 factorial plus z to the power 4 or 4 factorial minus z to the 5 by 5 factorial, and so on. All this stuff is divided by z, and then you can pull out this 1 over z outside and then you write down this as 1 over z minus the summation n equal 1 to infinity minus 1 to the n z to the 2 n minus 1 divided by 2 n factorial, right.

So, let us check if we got this sign, right. So, I have one, so it is just simply minus 1 to the n. So, if I put n equal to 1 and then I have z to the z to the 2 n minus 1 divided by 2 n factorial. Let us look at the first term. So, if I put n equal to 1, it is a minus 1. So, then you can have a z divided by 2 factorial which is maybe I have over connected for this.

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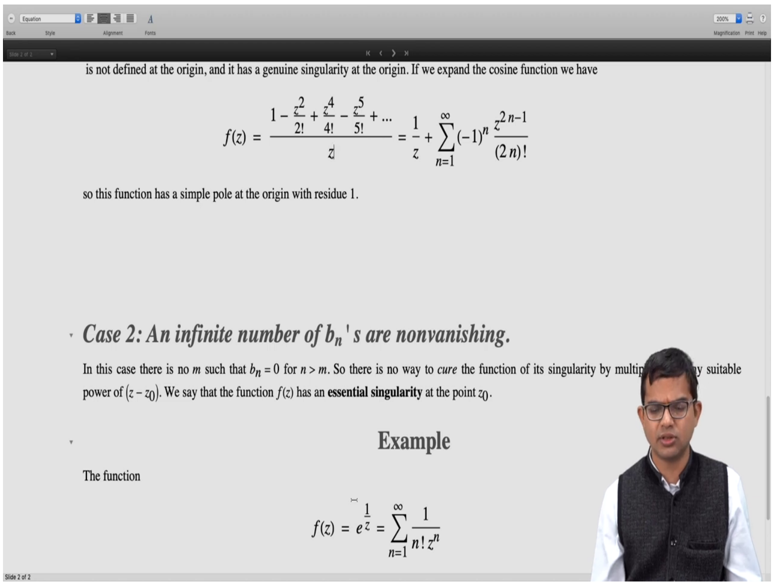

So, this should be just 1 over z plus 1 over z plus. So, then if I put an n equal to 2, then this is 3 z cube divided by 4 factorial which is good, and then if I have. So, n equal to 2, it comes with a positive sign. Yes, so this is correct. And so, this function has a, so this is the expansion for this function and it has a genuine pole and it turns out to be a simple pole because you see that it is just 1 over z.

And all of this is the regular part and the coefficient corresponding to 1 over z is just 1 and that is the residue for this. It is very simple, you know it is straightforward to evaluate the residue. But we will come back to residue residues a little bit later. Let us look at the third case and that is when you have all of these b n's are non-vanishing.

So, then you have these isolated singularities which are known as essential singularities. So, there is no way that you can multiply by some you know factor of some power of z minus z naught and you cannot cure this singularity and (Refer Time: 14:01) it, it is going to remain no matter what power of z 1 and z naught to multiply at this point. This is an essential singularity at this point

And so, the standard example for an essential singularity is this function e to the 1 over z, right. So, we have already looked at this kind of example. So, there is a singularity at the origin and it has this expansion 1 over n factorial z to the n. And so, no matter what power of z you multiply, there is going to be a singularity because all powers of 1 over z are there in this expansion. So, there is an essential singularity sitting at the origin.

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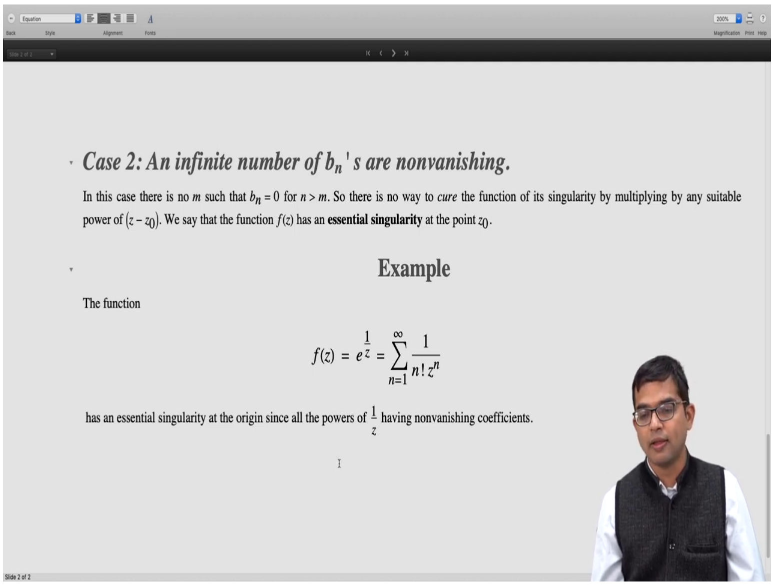

So, whenever you have an essential singularity, you know there are some, you know it has certain weird special properties around this essential singularity. So, that function tends to take basically any value you wish in its neighborhood, right. So, there are interesting properties of you know this weird kind of singularity which is our you know for very drastic kind.

But you know in general we are more interested in poles, and so, residues of poles and these kinds of things we will discuss in great detail. And so, that is where you know the power of the residue theorem which also is coming ahead, most of it is going to be actually connected to poles, right that is where our discussions are going to be centered now.

However, we should be aware that there are other kinds of singularities. So, one of them is of a trivial kind, then the removable singularity which was case 1, so this is actually case 3. So, that is case 2 and case 3. Case 2 is the standard kind. Case 3 is a very special weird kind which is also not very frequently encountered except in these kinds of special cases.

And then there is case 1, which is of a trivial kind. So, that is the 3 kinds of isolated singularities which we have looked at in this lecture.

Thank you.