Mathematical Methods 2

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Complex Variables
Lecture - 29
Derivatives of an Analytic Function

So, we have looked at the Cauchy integral formula. This powerful result about Analytic Functions and how the value of an analytic function at a point can be written in terms of its values along the whole contour around it right. So, and that contour can be quite far away from the point of interest provided the entire region is a region of analyticity.

So, in this lecture we will see how we can go further. So, in fact, the derivatives of an analytic function can also be written in terms of similar integrals. So, that is what we will discuss in this lecture ok.

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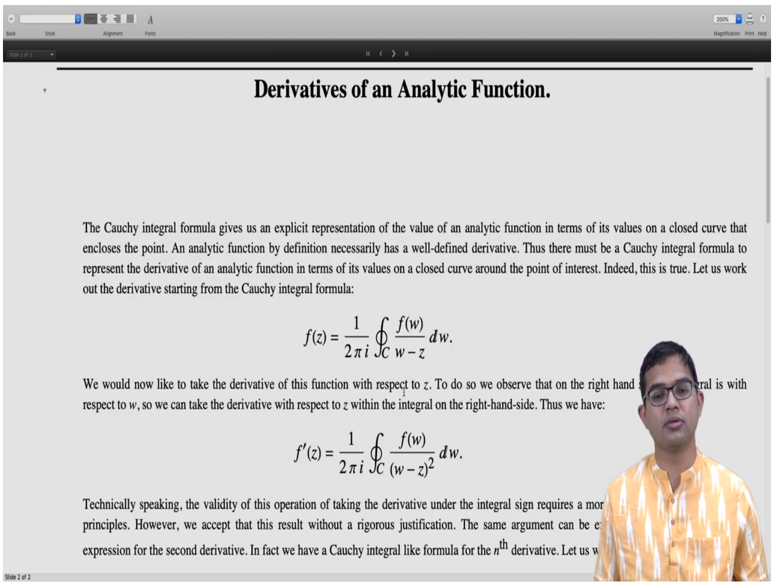

So, the Cauchy integral formula you know as we have seen is f of z while we wrote it as f of z naught right, but if you use the label z. So, f of z and then there is some dummy variable w.

You know you consider a contour C which encloses this point of interest, the point of interest being z and so we are given that this function f of z is analytic everywhere in the region enclosed by this contour including the points on the contour. Then we have seen that this contour integral 1 over 2 pi i contour integral C f of w divided by w minus z d w is in fact, the same as the value of the function f of z right.

So, now we see that f of z is a function of z and on the right hand side we have this you know complex variable w which it gets integrated out. So, in some sense it is a dummy variable right. So, we would like to take the derivative of this function f of z with respect to z right. So, we know that it is an analytic function. So, we are guaranteed that the derivative exists, there is a meaningful limit at that point.

And so, we are guaranteed that this derivative exists. So, we can actually go ahead and you know take this derivative like we typically would you know for this function 1 over w minus z right. So, the procedure is the so-called you know taking of the derivative under the integral sign right. So, we are probably familiar with this technique when we are working with functions of a real variable.

So, f prime of z is equal to 1 over 2 pi i. So, we have f of w will remain as it is, the only dependence on z within the integrand comes from this 1 over w minus z. So, you get minus 1 over w minus z the whole square times minus 1. So, it is just f of w divided by w minus z the whole square so. In fact, this is you know this is like a Cauchy integral formula for the first derivative of this function f of z right.

So, technically speaking this is you know the validity of this operation would require some more justification, but we will accept this. So, this is indeed correct and it can be you know argued for from first principles, but we will take this as a reasonable argument and indeed it is true f prime of z is given by this form, Cauchy integral like for it is another integral.

And so what we observe is f prime of z is also an analytic function. So, let us write this down formally first of all.

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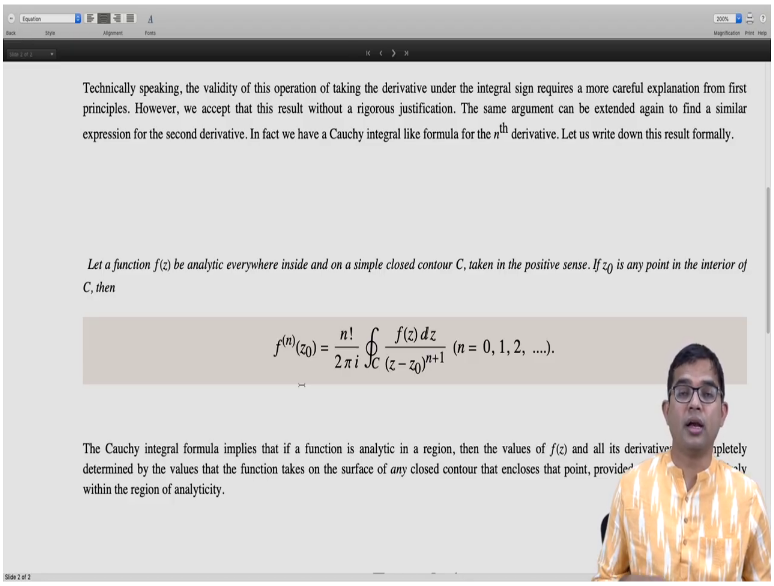

So, if you are given a function f of z; you know f of z is analytic and so, when you take that first derivative f prime of z is also analytic. And in fact, you can go ahead and take a derivative again and so if you do this same procedure repeatedly n times you can do it.

And so in general so, the result is that if a function f of z is analytic everywhere inside and on a simple closed contour C, taken in the positive sense and z naught is your point of interest which is interior; in the interior of C then we can find out the nth derivative of this function and the value of this at the point z naught is simply given by n factorial divided by 2 pi i right.

So, this factorial will come in because when you take a derivative once you it is just w minus z square, but if you take a derivative again then you will get 2 divided by w minus this is the whole cube then you will get a 2 times 3. So, in general if you take the derivative, you know as the nth derivative you are going to get an n factorial divided by 2 pi i as it is then the contour integral is now f of z divided by z minus z naught to the whole power n plus 1.

So, I have again gone back to older notation. So, I am looking at this point z naught and then n can be 0, 1, 2 and so on. So, n of n equal to 0 of course, refers to not taking any derivative that is the standard Cauchy integral formula.

So, what this tells us is that if a function is analytic in some region, then the value of not only the function f of z is determined by the values that this function takes you know along some contour right which is which encloses this and which you know encloses an entire region of an analyticity. But in fact, all the derivatives of this function at that point are given by the values of this function on such a contour right.

So, in fact a consequence is of course, is that the function is analytic then it is you know every derivative I mean higher order derivative must exist right. So, this is quite different from how you know this is the scenario in functions of a single variable right.

So, you may have a very nice looking function, but and who has a well-defined derivative, but the second derivative may not exist or a third derivative may not exist, but such a situation is impossible for an analytic function if its first derivative exists then second derivative exists, third derivative exists. And in fact, all derivatives to all orders are automatically present right. So, this is the extra power that analyticity gives you.

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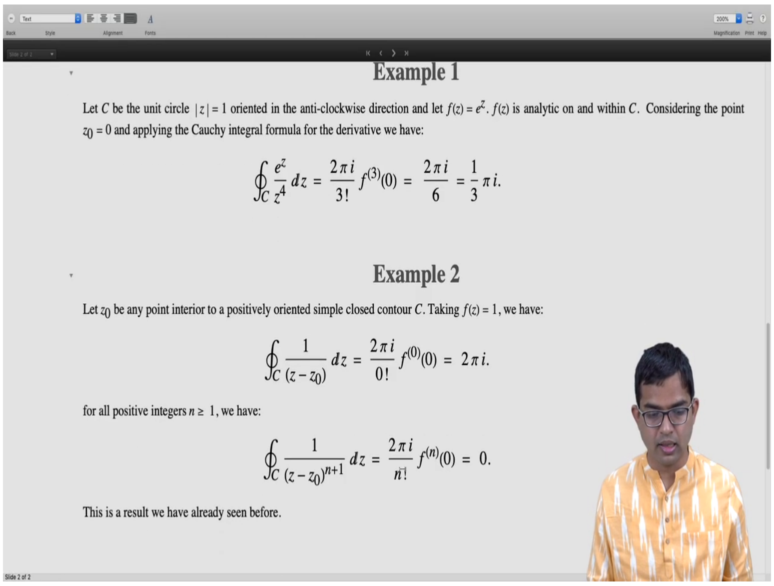

So, let us look at a few examples of how one can exploit this formula. So, suppose you have this unit circle mod z equal to 1 oriented in the anticlockwise sense and suppose you consider the function e to the z right, f of z is analytic on and within C. So, considering the point z naught equal to 0 and applying the Cauchy integral formula it is just a direct application of this result if you take e to the z and divide by z to the power 4 and you have dz.

So, z naught is taken to be 0 here right. So, and therefore, according to this formula you must get 2 pi i divided by 3 factorial and the value of the third derivative of this function evaluated at 0.

But the third derivative of this function is the same as the first derivative and is the same as the function itself because we are looking at e to the z. So, if you take the derivative of e to the z you get you get e to the z again and then repeated differentiation also leaves the function unchanged right.

So, this is exactly like how the function behaves even with the you know function of a real variable. So, therefore, you simply get 2 pi i divided by 3 factorial times e to the 0 which is 1. So, it is 2 pi i by 6 which is just 1 by 3 pi i. So, that is the answer, you can look at another example.

So, if z naught is some interior point you know that is enclosed by this positively oriented simple closed contour C and if you take the function itself to be just 1 right. So, it is a constant 1. So, of course, it is analytic. So, if you do 1 over z minus z naught right. So, this contour can be any simple closed contour right. So, we are not defining this contour you know with any greater specificity.

So, 1 over z minus z naught dz which is just 2 pi i 0 factorial f to the 0 of, 0 is the same as saying you do not do anything it is just this function as it is which is 2 pi i right. So, this is the result which we have already seen.

If you take 1 over z minus z naught dz contour integral about this point which encloses z naught is going to be 2 pi i and but if you take any higher order derivative right. So, 1 over z minus z naught the whole power n plus 1 where n is greater than or equal to 1 right.

So, then you are going to get 2 pi i by n factorial f the nth derivative of this function of 0, but even the first derivative of this function is just 0 and then of course, all subsequent derivatives are all 0. So, indeed this is equal to 0.

So, this is also a result which we saw right, we saw that the contour integral of 1 over z dz alone is 2 pi i, you know if you have positive powers also it gives 0 z to the n dz.

Whether n is a positive integer or a negative integer we saw that it is 0 unless you know except for this very special value 1 over z where n is equal to minus 1 right. I mean we basically are seeing the same result you know using this Cauchy integral theorem and of course, generalized to an arbitrary point z naught, it does not matter there is nothing sacrosanct about you know this result about the you know about the origin, it can be about any point z naught and that is what we have explicitly seen.

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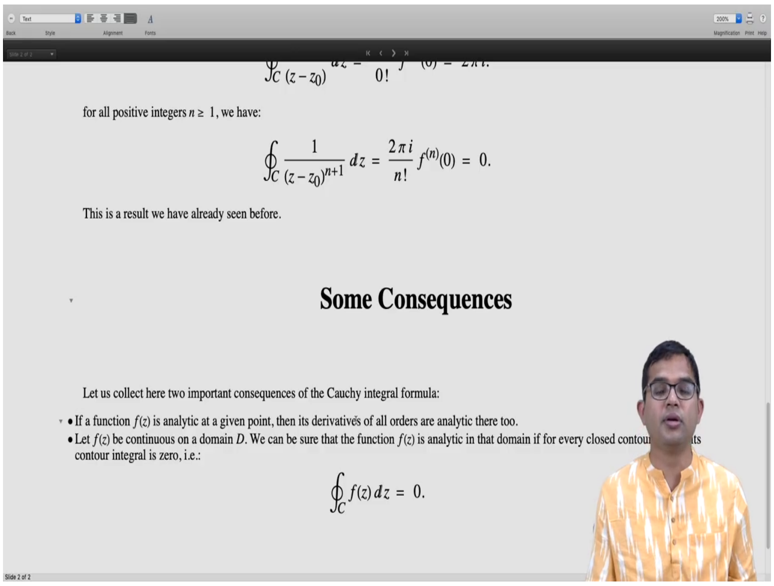

Ok, so, there are a few consequences which immediately come from this Cauchy integral formula for the derivatives.

So, let us just collect them here. So, if you have a function f of z which is analytic at a given point then all its derivatives; derivatives of all orders are also analytic right. So, this is something that I already mentioned. So, this comes from the fact that you have well defined derivatives right.

So, the existence of a derivative for f of z. So, if f prime of z is defined at a point right and we have seen that it is also the whole region, around this point is also analytic right.

So, therefore f prime of z by a similar Cauchy formula is going to be defined in this entire region around this point z. Therefore, f prime of z is not only is f prime of z it defined, but it is also analytic right because f prime of z is defined in a whole neighborhood and likewise every order thereafter is also going to be in an analytic function. So, you can keep on differentiating as many times as you want if you are working with an analytic function.

Now, it is also true that if f of z is continuous on a domain and if we can be sure that you know the contour integral, any closed contour integral of this function f of z is 0, then it is guaranteed to be analytic right. So, we have seen that you know this idea of continuity on a domain coupled with this need for every closed contour integral of the function being 0 is equivalent to the existence of an anti derivative right.

And, if the anti derivative exists and so, we can argue that it is basically analytic in this region right. So, these are the immediate consequences of the existence, the generalization of this Cauchy integral formula. So, for our purposes we want to look at analyticity and you know and all its properties without necessarily going into any rigorous proofs as such, but for the integral formula we did give a fairly convincing argument for how that comes about.

But in general so, the point is that if you have a function f of z which is analytic, it is going to be analytic in a region and derivatives to all orders are immediately available and all these derivatives are also analytic. And so, the statement that you know a function is analytic in some region is also equivalent basically to the statement that you know all these closed contour integrals of this function are also going to be 0 right.

So, which is also related to the fact that at every point f z you can associate an anti derivative to this function. So, it does not matter which path you take. So, this is also very closely related to the path independence of you know contour integrals which does not involve closed paths.

So, it can always be thought of if you are going from a point z 1 to z 2 along one direction. But then you should come back along a different direction then if you are guaranteed that this such a closed contour integral is 0.

So, basically what it means is you know going from z 1 to z 2 along one path must be equal to going from z 1 to z 2 along another path, because when the moment you have a closed contour integral which means that you know the path independence is immediately a consequence of you know every closed contour being 0 right. So, as you can immediately sort of intuitively see right.

So, in this lecture we looked at how the Cauchy integral formula can be extended to higher order derivatives and the fact that derivatives to all orders exist and are analytic for any analytic function.

Thank you.