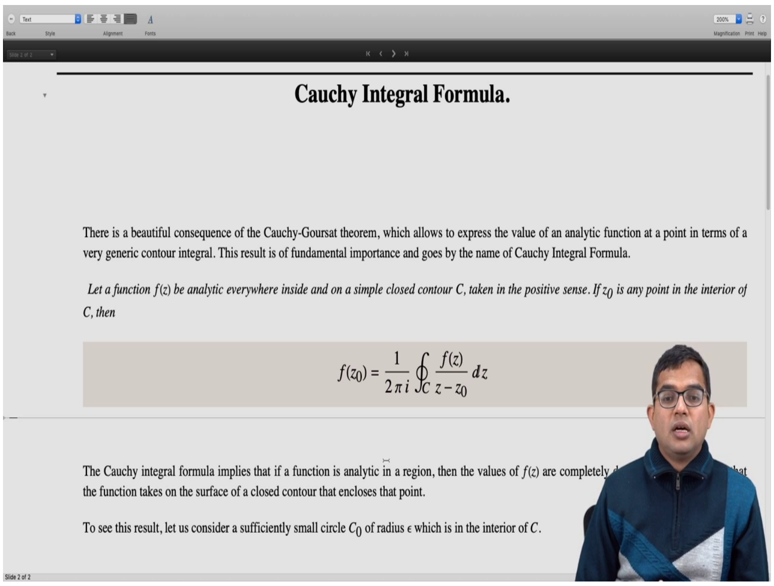
Mathematical Methods 2
Prof. Auditya Sharma
Department of Physics
Indian Institute of Science Education and Research, Bhopal

Complex Variables
Lecture - 28
Cauchy Integral Formula

So, we have seen how the Cauchy's theorem works out, and how the Cauchy-Goursat theorem which comes with less restrictive conditions on the function f of z says that if you have an analytic function f of z in some entire region, so in some region, and then you take a contour integral f of f of z dz where the contour passes the entirely inside your region of analyticity, then this counter integral must be 0.

And then we have seen how you know this idea can be extended to include multiply connected domains, we have seen how this can also be extended to consider contours which are not simple right. So, in this lecture, we will see a powerful consequence of the Cauchy-Goursat theorem, and this goes by the name of the Cauchy Integral Formula ok.

(Refer Slide Time: 01:10)



So, essentially what it says is if you are in a region which is completely analytic. So, you have a function f of z and you are looking at this function in a region where it's analytic. So, basically what Cauchy integral formula manages to tell us would give us a prescription for evaluating this function at any point in terms of the value of this function in a neighborhood around this right.

So, you can write down the value of this function as a contour integral over a path which surrounds it right which does not even pass through this point. So, in fact, somehow analyticity is a property where the value of your function you know at any point is connected to its values in the neighborhood right. So, we have seen this. And we will see other consequences also analyticity.

So, analyticity is a rather involved constraint. So, we have seen how the derivative if it is defined it must be defined in the same way regardless of the direction in which you take the limit, and somehow this has consequences for how you know the value of this function at any point is connected to its values in its neighborhood right, so that is that comes out in a very beautiful way Cauchy integral formula.

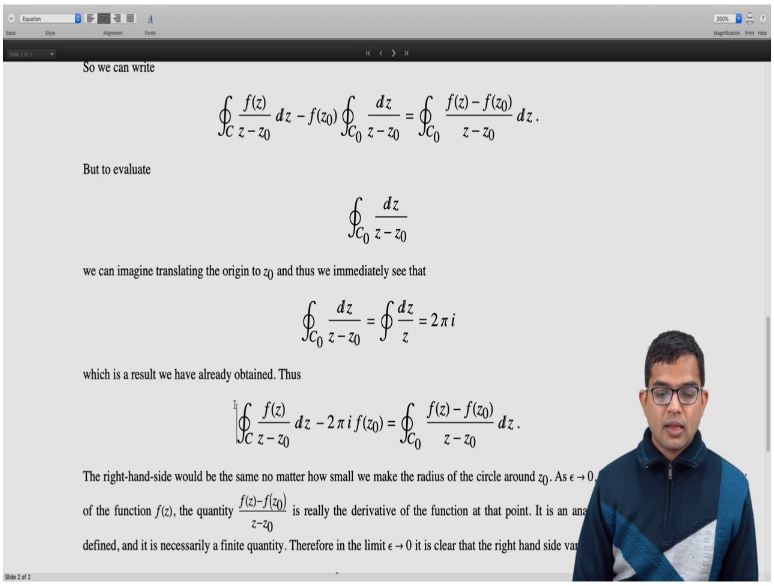
So, let us look at what the Cauchy integral formula is. So, if you have a function f of z which is analytic everywhere inside and on a simple closed contour C, taken in the positive sense. If z naught is any point in the interior of C, then f of z naught is nothing but 1 over 2 pi i contour integral over C in the positive sense f of z divided by z minus z naught dz.

So, observe that you know all that is happening on the right hand side is over a contour which is around this point. So, there is a way to get it you know the value of a function at a point z naught by looking at the same function itself in the neighborhood right that is what this integral formula tells us. And it gives you a very precise prescription for evaluating this. And this contour is actually quite general. It does not have to be you know very close by, it can be quite far away as long as it is within this region of analyticity ok.

So, let us see how this comes about. The Cauchy integral formula is actually a direct consequence of the Cauchy-Goursat theorem. And in fact, we will also make use of how you have the ability to deform contours right, so that is something that we will also look at. So, let us see how this comes about.

Suppose, we consider some small circle C naught of radius epsilon which we are immediately when we put something like epsilon we think of this as a small circle which we in fact take the infinitesimal limit of this in a moment.

(Refer Slide Time: 04:13)



But suppose you consider a picture. So, you have a picture like this. And so we want to consider this you know contour C as given in this you know the statement of the theorem. But suppose we connect it to a contour integral around z naught, z naught is the point of interest your function f of z is analytic at that point z naught. But suppose you consider a contour C naught which is a tiny circle right or circle of radius epsilon around this point z naught.

So, your function f of z is analytic, but you introduce this new function f of z divided by z minus z naught which is going to be not analytic at that point right. So, this function f of z is analytic everywhere inside you know in the region which is enclosed by C, therefore, at z naught for sure it is analytic.

But f of z divided by z minus z naught is not analytic at the point z minus z naught because that is your construction you have sort of artificially created a singularity at that point, but since f of z is analytic everywhere else so is f of z divided by z minus z naught its analytic everywhere except at this point z naught.

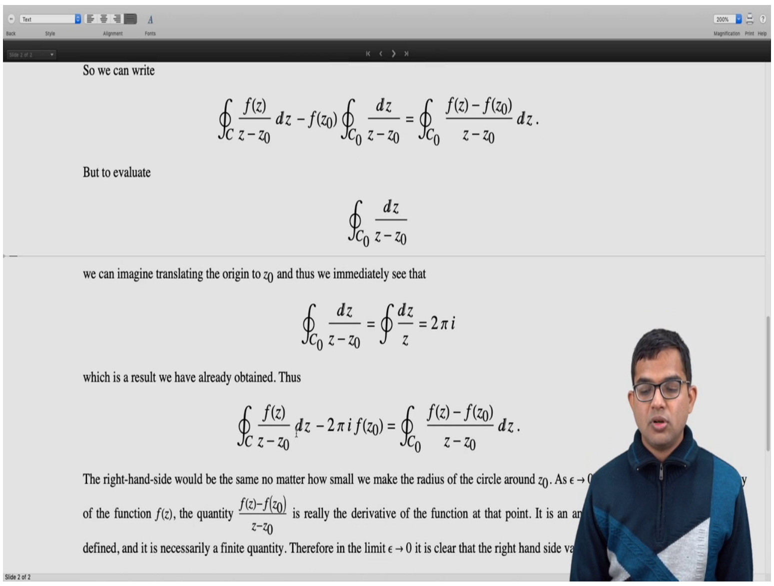
Therefore, by this principle of deformation of paths, we can immediately say that this contour integral over C of this new function f of z divided by z minus z naught dz is equal to contour integral over C naught f of z divided by z minus z naught dz right. So, we have in principle a very complicated contour C.

But what we are saying is that this contour integral f of z over z minus z naught over an arbitrarily complicated contour C is the same as this very simple contour right integral over this very simple contour C naught which is just a circle whose center is z naught and whose radius is epsilon.

And this radius epsilon we are thinking is very tiny right. Of course, we start with the assumption this epsilon is any number such that this C naught this entire counter C naught must necessarily lie inside c.

So, at no point can it cross C right. But we are imagining you know a process where we keep on shrinking this, but that is coming up a little bit later. So, what we first of all observe is that because you have this principle of deformation of paths, these two contour integrals are the same.

(Refer Slide Time: 06:52)



So, therefore, what we can do is we can write we can subtract this quantity the same number on both sides which is f of z times dz by z minus z naught right. So, on the right hand side, you see you have an f of z sitting here, but we basically subtract f of z minus f of z naught divided by z minus z naught.

So, and f of z naught is just a constant, so that will come out, and then you are left with this contour integral dz by z minus z naught right. So, this is, all I am doing is subtracting the same quantity on both sides. So, this quantity minus this quantity is equal to you know I just write it as f of z naught in the numerator here on the right hand side. So, f of z minus f of z0 divided by z minus z naught dz.

Now, this integral, this contour integral that I have subtracted on the left hand side is something that we can work out, in fact, we have already worked this out right it just we have to look at it carefully. So, what is this quantity? So, I have a point z naught and I am looking at 1 over z minus z naught and finding a contour integral about a circle around which is centered at z naught right.

And, but this is really the same as you know doing dz by z over a contour which is centered about the origin right which is something which we have already done right. So, it is a matter of the symmetry principle right whether you know put a singularity like 1 over z at the origin, or you put the same kind of singularity at another point 1 over z minus z naught and consider a contour which is in the same positive sense about the point z naught or about the origin, it is the same right.

So, you can do a sort of translation of your origin or a shift of the origin. And so therefore, the results without doing any calculation is that this answer is just 2 pi i right. You can also do the calculation explicitly and check this. So, the way to do that is of course, to put z is equal to z naught plus you know some epsilon e to the i theta, and then carry out this integral over theta right, so that you will anyway get back the same answer.

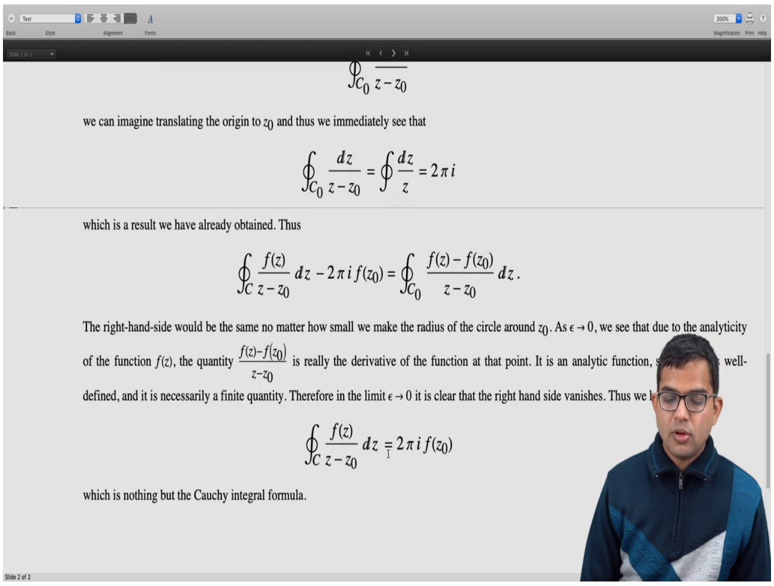
So, the answer is 2 pi i in this case. So, if we plug this result back in here, so what we have is contour integral over C f of z divided by z minus z naught dz minus 2 pi i times f of z naught is the same as this contour integral over C naught, C naught is this you know tiny circle around the point of interest z minus z naught f of z minus f of z naught divided by z minus z naught dz.

So, now we will take the right hand side and keep on shrinking this. And in other words, we are going to take the limit epsilon going to 0. And then we see that this quantity is actually nothing but it is like a you know take like taking a derivative of this function, it is an analytic function. And so since it is an analytic function this limit is well-defined. It does not matter in which direction you approach it is going to be the same and it has a finite value right.

And we are multiplying this finite value with an infinitesimal quantity and taking the limit going to 0 and this is going to be 0 right. You are guaranteed that this quantity is 0 as you take the limit epsilon going to 0. And therefore, it is something you can just put it to be 0 on the right hand side right. So, it has to be the same no matter what the value of epsilon is right.

So, this is you know this is the principle of deformation right. It does not matter what the value of epsilon is, you have got to get the same answer. And therefore, you can take the limit of epsilon going to be very very tiny. And since it vanishes, this integral is necessarily 0 on the right hand side.

(Refer Slide Time: 10:54)



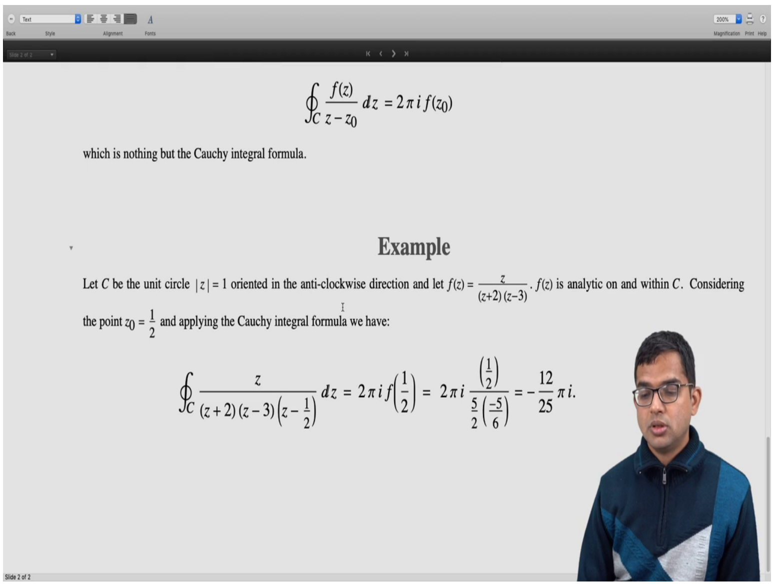
So, immediately from this we have the result that you know integral f of z divided by z minus z naught dz over this contour C which encloses z naught is going to be 2 pi i f of z naught. As long as this contour is you know passes through an analytic every point on this contour and all the points enclosed by this contour, your r analytic as far as the function f of z is concerned.

So, the Cauchy integral formula, let us look at it once again tells us that f of z naught right, so the value of this function f of at any point z naught is simply given by you know a contour integral of f of z divided by z naught or z minus z naught. In some sense what you have done is you have divided by z minus z naught and introduced a sort of artificial singularity at this point z naught as far as this integrand is concerned.

And then you have cleverly integrated this function around the contour. And then managed to show that in fact f of z naught is given by stitching together the values of this function in the neighborhood can get back to you the value of the function itself at this point. So, it is a very beautiful result. And it has many useful applications.

And the ideas which go into making this result hold also have many applications. So, it is worth understanding it well. So, this is the Cauchy Integral Formula.

(Refer Slide Time: 12:20)



Let us look at an example where we can apply this. Suppose, we consider the unit circle mod z is equal to one oriented in the anti clockwise direction and consider some function f of z you know you can cook up your own function of interest z over z plus 2 times z minus 3. So, notice that the you know singularities of these functions one is at minus 2, and the other is at 3 – both of them lie outside this unit circle. So, there is no issue.

So, f of f of z is analytic on and within C, therefore, the Cauchy integral formula holds. So, at random you can pick any point within this region which is enclosed by mod z is equal to 1. Suppose, we pick z naught equal to half and then if we apply the Cauchy integral formula, then we have this integral contour integral of z over z plus 2 times z minus 3, then we also have to divide by z minus half because I have picked z naught to be half, then this is going to be equal to 2 pi i times f of half right.

But f of half is half divided by half plus 2 which is 5 by 2, and then half minus 3 which is minus 5 by 6. Then if I do the algebra carefully, it is just minus 12 over 25 pi i. So, you should check this right.

So, this is just an example where we directly plug it in, but you have to be careful about choosing your region of analyticity carefully if your function has to be analytic entirely on this contour and all points which are you know enclosed by this contour the Cauchy Integral Formula holds. That is all for this lecture.

Thank you.