Mathematical Methods 2
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Module - 02
Complex Variables
Lecture - 21
Branch of a multivalued function

Ok. So, we have looked at a number of examples of functions which are multivalued, right. Many of these came about when you know we tried to find inverses of functions which were single valued. So, whenever you have a several you know inputs basically take you to the same output, it is not a one to one function, and if you try to invert such a function then you will be in a scenario where the same value of z is bound to multiple different values for the inverse function.

So, let us take a closer look at such multivalued functions in this lecture and see how to deal with you know this multivaluedness, ok.

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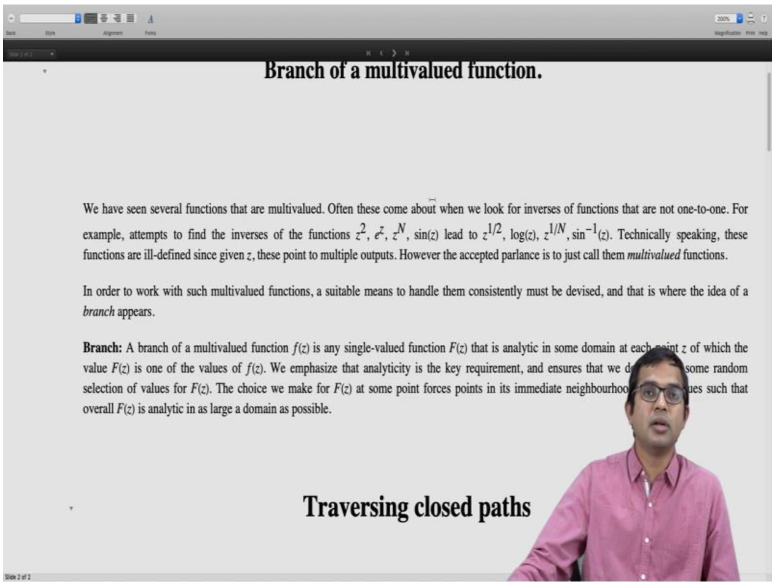

So, yeah so, examples of multivalued functions and how they appear, you know when we look at inverses of say functions like z squared, right, if you try to invert z squared then you have z to the half, and yeah. So, we know that both plus z to the half and minus g to the half when squared will give you z squared. So, there are two different you know possible answers for any given z.

And likewise, if you look at e to the z we have seen that log of z is in fact, there are infinitely many values of log of z, right. So, we can associate with log of z. In the sense that, if you take the exponential of all of these different numbers you will get back the same e to the z. And similarly with z to the N in general, where N is some integer, so if you try to invert this, so you have z to the 1 over N, right and we have seen that the Nth roots of unity there are N of these, right.

So, in general z to the 1 over N, there are going to be N different values which correspond to z to the 1 over N. And likewise, also sin inverse of z, we have seen is you know connected to the log function and if since the log function itself is multivalued, sin inverse of z is also multivalued, and well also the square root multivaluedness also appears there, right.

So, technically speaking you cannot really call this as functions because the technical definition of a function is that given an input there should be a unique output otherwise it is not a function. But you know it is accepted parlance to just call these as multivalued functions and come up with a you know practical way to deal with such functions.

So, one might think that you know you think of all these different possible values of these functions as representing separate functions, right that could be an idea. And so, that is connected to the idea of a branch, right. So, you think of different branches, right. Instead of having just one complex plane you think of many it being made up of you know what are called Riemann sheets and so, then you might think that this is going to solve the problem.

So, suppose you are looking at something like z to the half plus z to the half could be one function and minus z to the half could be a separate function, right. We could try to solve this problem by looking at them as two different functions or you know which are usually described as two separate branches. But this itself does not quite completely solve the issue as we will see because there are certain topological aspects as well which we should keep track of.

So, let us define a branch. A branch of a multivalued function f of z is a single-valued function. Inside a branch a function becomes single-valued F of z like we have already done it with the log function. So, small log of z is multivalued, but we looked at capital Log. So, in addition to it being single-valued we would also like it to be analytic, which was a very important property that we want to retain. So, we do not want to assign you know unique values to capital F of z, but in some arbitrary way. If you do this then you are not going to keep the analyticity of F of z, right.

So, suppose I am looking at z to the half, right, I can have either plus z to the half or minus z to the half, both of these are valid outputs, but I cannot assign you know plus and minus in some arbitrary way, right. So, it should stick to all of them being minus or all of them being plus when I am looking at you know neighbourhood of a point as well counts that were interested in the idea of analyticity, right.

So, we would like to make this function single-valued, but also analytic in as large a region as possible, right. So, that is why it is a subtle aspect, right, how to treat multivalued functions, right. So, we must ensure that F of z behaves in a nice way in a neighbourhood around the point of interest not just at that point, ok.

So, let us look at an example, suppose, yeah. So, this example will illustrate how; it is not enough to just consider you know separate branches, it does not quite solve the problem, the reason is the following, right.

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So, let us look at a simple example. So, we will consider the function f of z is equal to the argument of z, right. We have seen that the argument function is multivalued. So, given you may be at a certain point z, but there are you know either you can represent it with a certain angle theta or theta plus 2 pi is an equally valid angle to represent that point in polar coordinates, right.

So, this function; so, what happens if you start from a certain point, you assign a certain value and then you slightly move around, you go around a closed path. Now, the question is does the function, when you come back to the same point, is there a way even if you are in a certain branch will you return to where you started, right? So, if you ask this question; so, let us try to work this out for f of z and consider two different kinds of loops, right. So, let say, ok.

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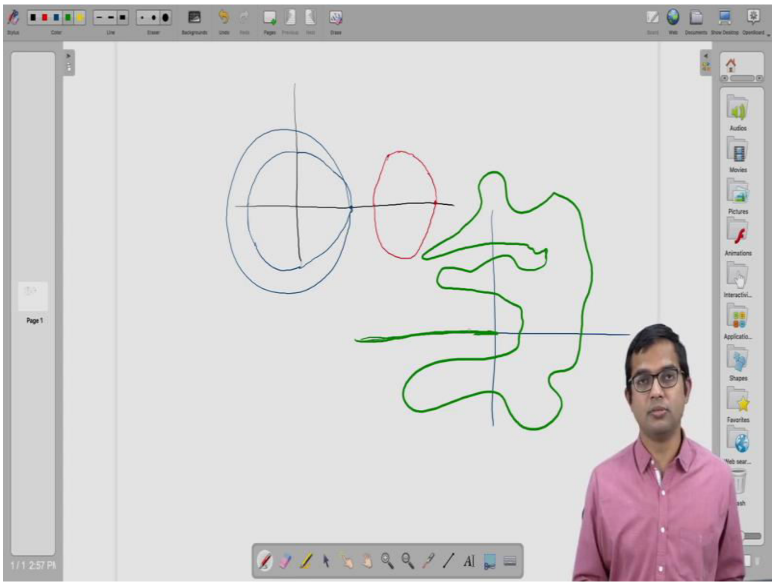

Let us look at; so, this is the complex plane and then let us say I start with a certain point. Let me use a different colour for this. Suppose, I take this a path where I start with from a certain point and then I go up. So, when I am here, I assign the value theta equal to 0, right. So, and then I start moving up. You know slowly it increases, it increases, keeps on increasing and then I reach a certain point where I am I am supposed to measure the angle with respect to the origin, right.

So, if I were to draw a straight line from here, you know you can imagine a straight line from the origin all the way to you know let us say this point and the angle that you know yeah this line makes with the x axis, would be the angle at this point, and then it keeps decreasing.

As you come down, you see that it keeps decreasing. And then when I come back to this point when I hit the x axis again it is going to be 0 again, right. You can start it from 0, it increased to some positive value, high positive value, which represents which could be this angle I do not know maybe something like 30 degrees, 40 degrees whatever in this case, then it comes back down to 0.

And then, if I go around and then if I come back to this point that is what that is the type of path that I am thinking of, right. So, then I go; so, now, it is in the negative direction it goes back in the negative direction and then I come back to where it is started and then it is again it has come back to 0, right. So, it seems like there is no issue.

But on the other hand, suppose I do the same kind of an exercise, but for a different closed path. So, let me use a different colour here. So, this is path b. So, suppose I start at this point and then I start moving up, so the angle increases, angle increases, and then I cross the imaginary axis and then come around.

So, you see that the angle at this point is actually pi, right. So, I have not restricted you know argument of z to have only certain values at this point. I am looking at the function small arg of z. So, if I continue to you know go beyond this point, so it is going to increase. My you know angles will be 5 plus something, so on, and then at this point it is going to be 3 pi by 2 and then I come back to this point, by the time I hit back this point just come to 2 pi, ok.

So, what we find is if I take this second kind of a path, you know I start from this point on the x axis and then return to it in a closed loop, and then when I come back to where I started the value I ascribe to the function arg of z. I am at the same point has it has changed by 2 pi. And if I were to do one more loop like this, suppose I go around and then if I do one more loop, then now the angle is going to be 4 pi and if I do one more loop it is going to be 6 pi and so on, right.

So, you know it is as if you although you started in a certain branch, right, if you go around it is you have to have a way to certain kinds of loops, you have to ensure that you know actually entered a different branch, ok. So, it is not enough to be able to think of a multivalued function as being made up of you know several different functions which all operate in different branches, but they have to be connected in this kind of a special way, right. So, that is what makes this you know multivalued functions subtle, right.

So, we observe that whenever the origin lies inside such a loop then you are going to get you know a different value for such a loop value, from you know the value that the function has at the starting point and the value that the function will take at the end point are going to be different. But on the other hand, if the origin we have to lie outside the loop then we saw an example where there was no issue, right.

So, in some sense you are all you are all part of the same branch, if you are you know if you are not, your loop does not include the origin, right. So, we must be able to use a you know tool by which we can separate out these two kinds of paths. And so, it is not enough to introduce the notion of a branch, but we also need to introduce something called a branch cut.

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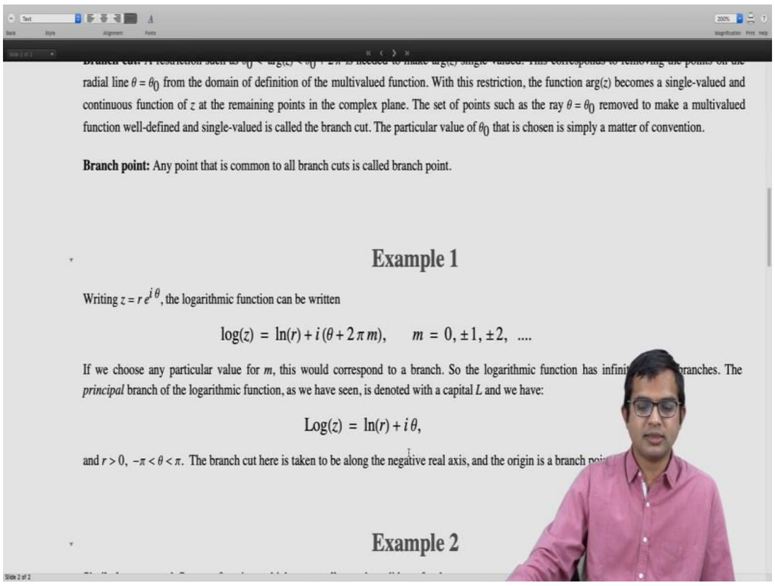

So, we must put a restriction that your angle theta naught you know must lie inside some you know set of angles which goes from theta naught to theta naught plus 2 pi. You have the freedom to fix your theta naught, but we must remove a whole ray of line ray of points, right. So, this corresponds to removing this radial line theta equal to theta naught from the domain of definition of the multiple valued functions. So, you this is one way to solve this problem.

So, another way is to use something called Riemann sheet, which, let us not go there at this point. So, let us think of you know a single branch and how we can introduce a branch cut, and then we can maintain analyticity of you know multivalued functions in the entire branch except for that one line although except for this cut, right.

So, if you make this restriction then the argument of z becomes a single-valued continuous function of z and at all the remaining points in the complex plane it is actually analytic, right. So, this set of points such that you know such as this ray which is removed to make it a well-defined and meaningful single valid function is called a branch cut. So, it should be set of points.

Now, a branch point is any point that is common to all the branch cuts, right. So, we could have, you know you have many sort of a stack of different branches in some sense, but they are all you know connected at this point and along the branch cut, in fact, right. So, that is the picture that one uses when thinks of something called a Riemann surface.

But at this point let us concentrate on just a single branch and the presence of this thing called branch cut. And practically, we can do all our you know we can go ahead and you know study analyticity, and you do all the nice things that analytic functions have as long as we do not cross this branch cut.

So, let us look at a few examples. So, suppose we write r is z equal to r times e to the i theta. So, the logarithmic function is multivalued in an extreme sort of way because it has the for any given z there are actually infinitely many values that this function can take. So, log of z is equal to log r plus i times theta plus you have this freedom to add i times 2 pi m, where m can be any integer, right.

If you choose a particular value of m, then that would correspond to one branch. So, it has actually infinitely many branches. And if you choose any particular m, right, when you are dealing with log z and in a certain branch then you know it has its analytic within that branch as long as you do not cross the cut. So, and one special branch that we typically deal with is the so-called principal branch, right, which we have already looked at, and so, that is defined with a capital L.

So, log of z is equal to log r plus i theta, where you put m equal to 0, and then you have you know logarithm of any complex number in this principle branch is simply given by you know the Napierian period logarithm of the radius that is mod of z plus i theta, right. So, theta is restricted to be to lie between minus pi and plus pi.

So, here, in fact, we are making use of you know the capital Arg, the argument is also restricted to lie between minus pi and plus pi. So, in fact, we have introduced a branch cut here, along the negative real axis starting from the origin, which is actually a branch point, right.

So, branch point is the point where all these you know different branch cuts, they all interact, it is common to all the branch cuts, right. So, in some sense, if you take all these different branches, you have to sort of stick them all together onto this branch and they all and they all meet at this branch point in the Riemann surface picture, ok.

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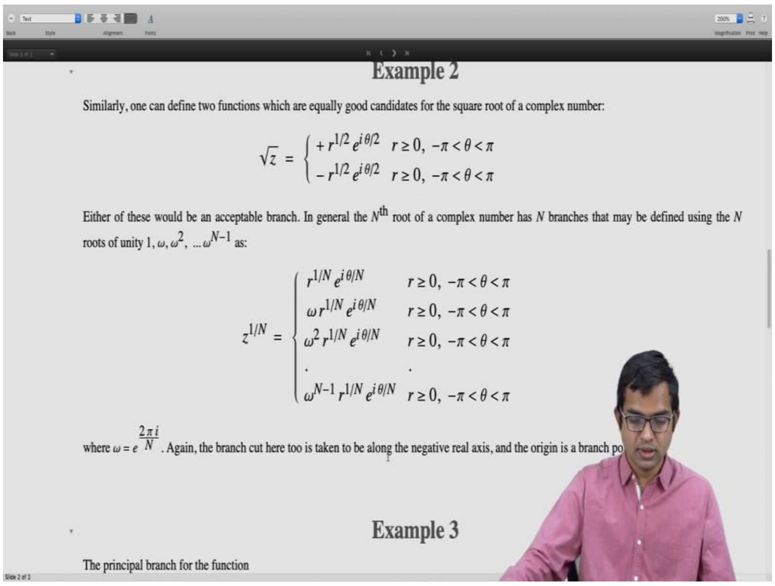

So, let us look at another example. Similarly, one can define suppose we looked at the square root function. So, square root function has these two different branches, right, either of these values would be equally good candidates. You could either take plus r to the half times e to the i theta by 2 with r greater than or equal to 0 or you could take minus r to the half e to the i theta by 2 with r greater than or equal to 0. So, here of course, r equal to 0 is this logarithmic singularity, right; so, the in the earlier example. So, it is not defined at that point the log function. Whereas, here there is no problem.

So, a branch point could be you know could be a benign one like here in this case square root of z also has a branch point at the point r equal to 0, but it is just 0, right. So, you know there are different forms that a branch point may appear in. So, in this case there are just two branches. So, plus branch and minus branch if you wish.

And so, in fact, you could have you could do this kind of an exercise where we did it for the argument function. So, if you take you know if you take the positive branch and then go around in a loop around the origin, if you were to cross the branch cut, then you would actually end up in the next branch.

So, that is why even for you know to ensure that you stay within the same branch you must impose this branch cut and you know if you are taking any closed loops are not allowed to cross this branch cut and then you remain within the same branch. So, in this case, I have already restricted theta to lie between minus pi and plus pi. So, again taking you know argument to lie between minus pi and plus pi, right. So, theta equal to pi itself is not included.

So, either of these would be an acceptable branch. And so, in general the Nth root of a complex number has N branches as we have seen, right. So, we could write these N branches in terms of the end roots of unity. So, we have one omega, omega squared, all the way up to omega to the N minus 1 are the Nth roots of unity, so we could define z to the 1 over N. You know there are these N different branches. The value that your function will take if you are sitting in each of these different branches is given, right.

So, in some sense, what we are doing is we are converting a multivalued function into many different single-valued functions and introducing the idea of a branch cut, so that you not only have single-valuedness, but also analyticity in you know as larger region. So, if you stick to one particular branch and you do not cross the branch cut, then your function is very nice and well behaved single-valued analytic all the nice things are available for you, right.

So, again the origin is a branch point here and we have chosen negative real axis. So, we said that in general you can choose the branch cut to be any ray starting from the origin going to infinity along any direction, right. As long as you stick consistently to your branch cut it does not matter you can do your calculations in you know whatever you know the branch cut is and the convention that we are following here is to take this branch cut to lie along the negative real axis starting from the origin.

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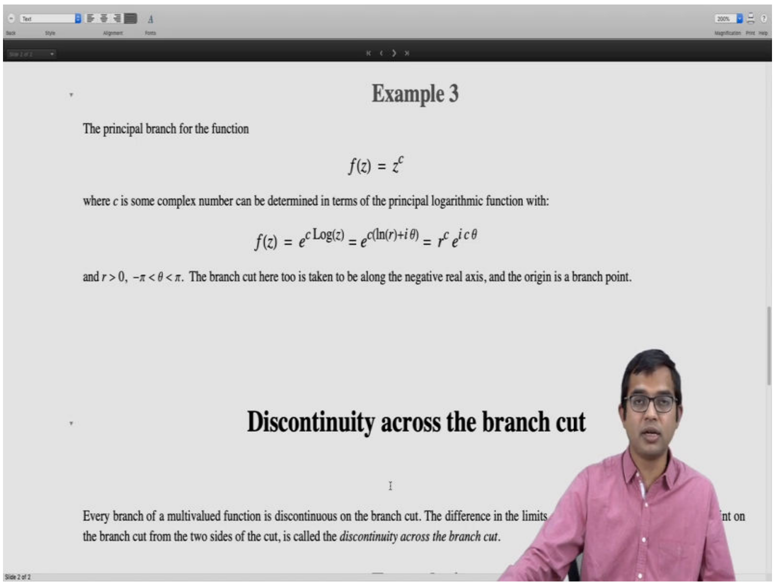

So, let us look at another example. So, if you have f of z is equal to z to the c, where c is some complex number. So, in general, if you have z to the 1 over N, as N different branches, so you will also have N different branches if you consider z to the 1 over N to the power some integer, right. So, that would be of the form you know z to the power of a rational number.

So, what appears in the numerator does not really matter. So, it is the denominator. So, 1 over N or you know some other integer have capital M over N. So, z to the capital M divided by capital N, where M is some other integer and N is another integer, it is this function also will have N branches.

Now, but on the other hand, if you look at a function like f of z is equal to z to the c, where c is a complex number, right, you if you do not you know it is neither an integer nor a you know the ratio of two integers then this function is going to be definitely it is multivalued. But, in fact, it has infinitely many branches because you have to define this in terms of the log function and the log function has infinitely many values, right.

So, we have seen that z to the c is defined as e to the c times log of z. And if you stick to the principal branch then you can define this f of z to be just r to the c, r is a real number to the c. So, there is no problem with that, right. So, that is understood in terms of the Euler formula and then you have e to the i c theta. So, the whole point here is that we are restricting this angle to lie between minus pi and plus pi. And once again the branch cut is taken to be along the negative real axis and the origin is a branch point.

Now, when you have a, when you introduce a branch cut, so the function is not going to be analytic you know along the branch cut. And, in fact, there is a sharp discontinuity across the branch cut, right. So, let us look at how this plays out. And this is called the discontinuity across the branch cut, right.

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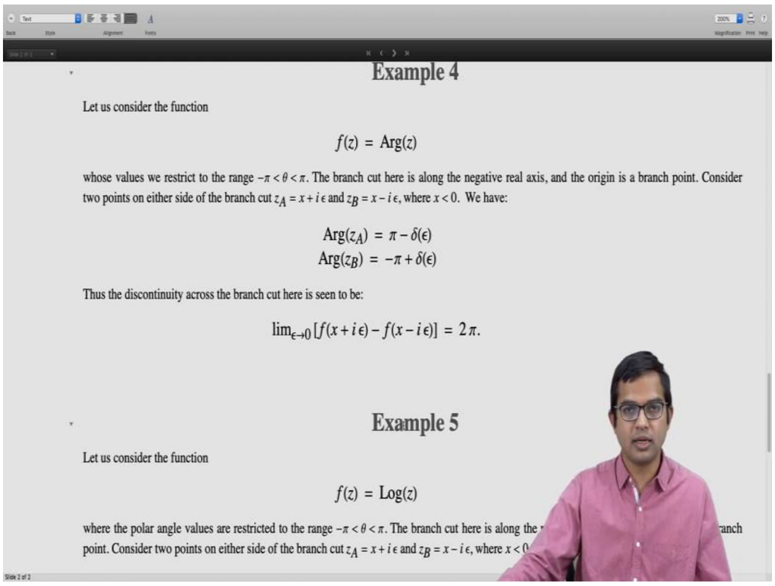

So, every branch of a multivalued function necessarily has you know this discontinuity across the branch cut. Yeah. So, I mean the reason this happens is in some sense all of these different branches are glued together along this branch cut. Somewhat hard to imagine, but you know you can think of a stack of different branches, but they are all you know they share this branch cut and you are not allowed to cross it, right, if you want to stick to the same branch.

But actually what happens is if you do cross it you enter some other branch, so that is the picture of a Riemann surface each of these sheets they are called Riemann sheets. And then, you cross you go around once if you cross the branch point in one direction you enter a different branch and there is a way to do the bookkeeping for this.

But suppose, we stick to a single branch, right and then if you try if you cross the branch within the same sheet then you have a discontinuity. But if you were to enter a different you know the suitable one, then you know if you enter the right branch then it is not going to be discontinued. So, that is the picture of the Riemann surface which we will not look at this point.

Suppose, we stick to a single branch and then you try to make a if you cross this branch, so this is what some function; let us look at an example where this discontinuity is something we explicitly work out. So, we go back to the argument function. Let us say you stick to the principal branch. So, we define f of z with a capital A, so Arg is capital Arg of z, and then whose values we restrict to lie between minus pi and plus pi, right, theta equal to pi itself is the is a branch cut.

And so, suppose you consider some point which is sort of infinitesimally you know away from the branch cut you know slightly to the positive side and then there is another point which is slightly below the negative real axis. So, we consider these two points z A is equal to x plus i epsilon and z B is equal to x minus i epsilon, and we look at the case where x is less than 0.

Then, we have an argument of z A and slightly above this point is going to be pi minus some delta of epsilon, right. So, there is an angle associated with this small epsilon that you have moved up, right. You can compute it. But it is not important what the precise values is that some small value which is you know you have to subtract, so it is slightly less than pi.

And on the other hand, the angle which corresponds to the point z B is minus pi plus delta of epsilon. So, if you were to compute this limit in the limit of epsilon become very tiny. What is the difference between the functional values? Slightly above the branch cut and minus slightly below the branch cut, then you see that it is in this case f of x plus i epsilon minus f of x, minus i epsilon is just given by pi minus delta of epsilon minus minus pi.

So, this delta will anyway go to 0 as you take the limit epsilon going to 0, it is only this 2 pi that will remain. So, there is this discontinuity 2 pi, no matter where you are. So, it is independent of you know how far away from the origin you are for this particular function, but it is a constant discontinuity of 2 pi.

Let us look at another example. Suppose, we consider the function log of z, right which is our favourite multivalued function. And now, once again if we do the same kind of an exercise.

So, we restrict the polar angle to lie between minus pi plus pi. Branch cut once again is along the negative real axis starting from the origin and going all the way up to minus infinity.

And we consider these two points z A is slightly above, the negative axis z B slightly below and then we have to compute the value of the function slightly above and slightly below. So, argument of z A and z B as we already seen are pi minus delta and minus pi plus delta.

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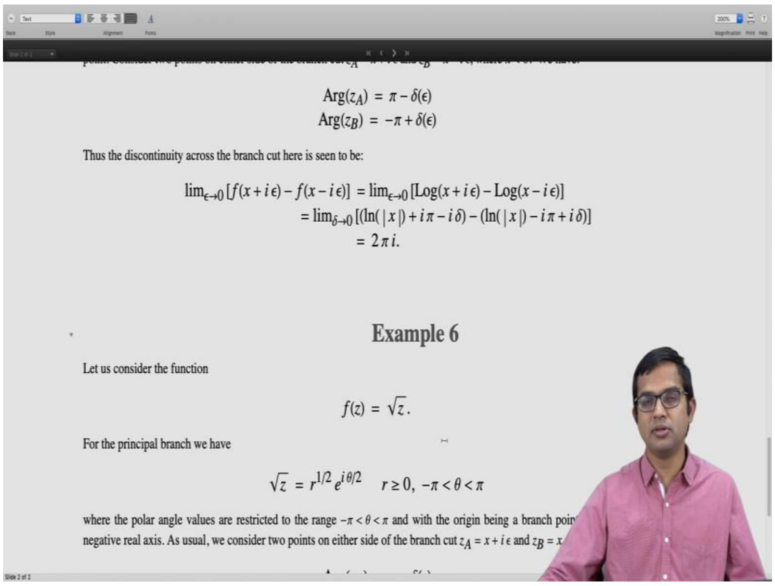

So, in this case the discontinuity we can work out is given by f of x plus i epsilon minus f of x minus i epsilon will be log of x plus i epsilon minus log of x minus i epsilon. So, we have seen that you know the way to compute this quantity is to just find the you know the Napierian logarithm of the mod of x which is just a you know regular a logarithm value of a positive real number plus i pi minus i delta because that is the angle, that is the argument of this quantity which is just pi minus delta. So, you have to do i times pi minus delta. And, but on the negative side you have minus log of mod x is the same, but now you have minus pi plus delta, right. So, plus i times minus pi plus delta.

So, these two will cancel. So, it is a pi minus delta minus i times, yeah; so, this is correct, minus i pi plus i delta. So, the in any case the delta part is going to go to 0, as epsilon goes to 0 delta also will go to 0. Log mod x and log mod x will cancel, and then you are just left with i pi plus i pi which is 2 pi i. So, once again we see that even for the log function the discontinuity across the branch cut is independent of how far away you are from the branch point, ok.

So, let us look at another example. So, square root of z, right, for the principal branch we have square root of z we can take it to be the positive value. So, r to the half times e to the i theta by 2 and r is greater than or equal to 0, minus pi lies theta lies between minus pi and plus pi. And once again we consider z A to be slightly above the negative real axis z B to be slightly below, and then the arguments of you know these two points, we already worked out pi minus delta and minus pi plus delta.

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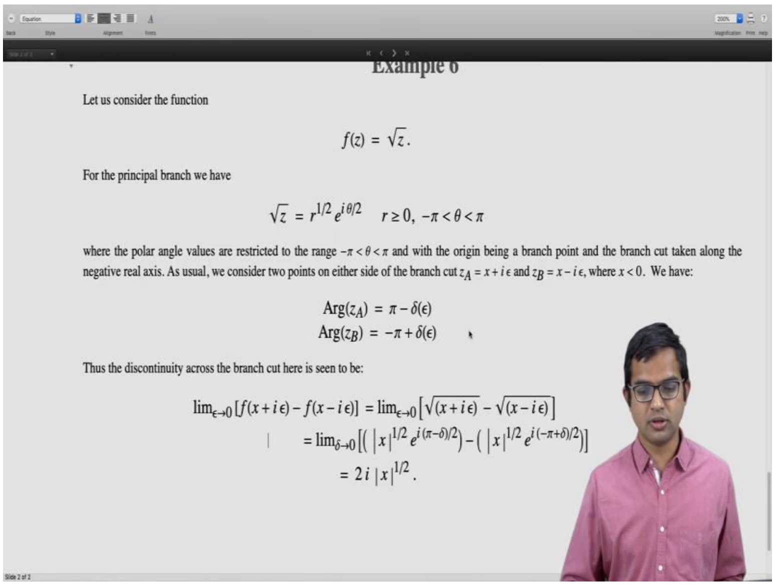

So, if we compute the difference in the value of this function at these two points, so we see that one of them is square root of x plus i epsilon and the other one is square root of x minus epsilon. And so, the way to compute it is according to this formula, right. So, I am just simply using the value of the angle at these two points.

So, mod of x to the half times e to the i pi minus delta by 2, then we have a minus mod of x to the half times e to the i minus pi plus delta whole thing by 2. If we combine this carefully, so we will see that the answer is just; so, it is the mod x to the half comes out. And then we have a cosine; well, I mean we can simply put delta equal to 0. So, we have e to the i pi by 2 minus e to the minus i pi by 2, so which is nothing but 2 i sin pi by 2 which is just 2 pi, right. So, this is something that you can check, right.

So, we have looked at a few simple examples, but you can consider some other more complicated multivalued function and try to compute the discontinuity across the branch cut. So, once again I emphasize that this discontinuity appears if you stick to the same branch, right. So, in some sense what you should do is, if you have if you have a the complex plane and if you have made this your forbidden region in some sense you have chosen this entire region is your branch cut.

So, if you are going to look at the properties of your function, as long as long as, as long as you move around like this, and then you it is not a problem, right. Anywhere here, your function has all these nice properties, it is analytic, and all this it is only here that you are not allowed to cross, right.

So, we will return to this at a later time. But at this point since we looked at, so many multivalued functions, it is useful to complete this discussion about how to deal with multivalued functions and how to make them single-valued effectively inside a single-valued branch as it is called. So, and the principal branch being a special single-valued branch, ok. So, that is all for this lecture.

Thank you.