Mathematical Methods 2
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Module - 02
Complex Variables
Lecture - 19
Hyperbolic functions of complex variables

So, we have seen how we can generalize trigonometric functions to allow for Complex Variables. So, in this lecture, we will see how in a similar analogous manner, we can use the generalized idea of the exponential function to also define Hyperbolic Functions, we will also look at some of their properties, ok.

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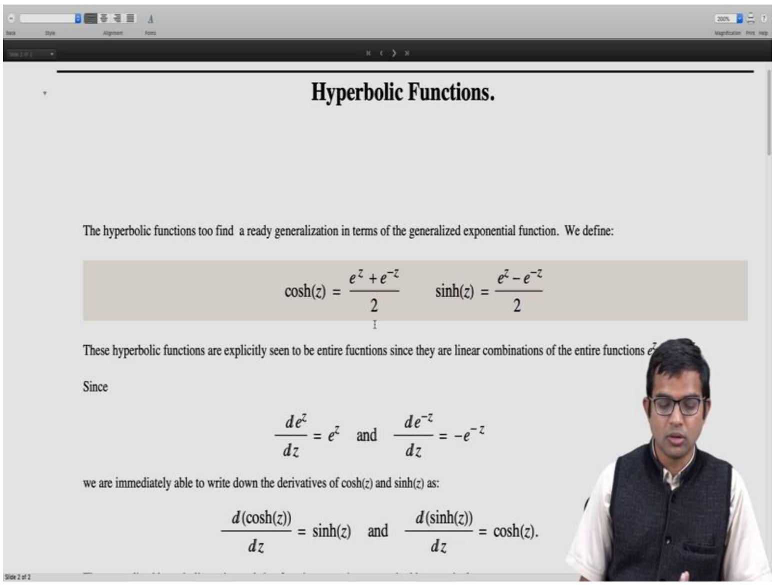

So, we define hyperbolic cosine function of a complex number z as e to the z plus e to the minus z divided by 2 and the hyperbolic sine function of a complex number z as another complex number, which is defined as e to the z minus e to the minus z the whole thing divided by 2, right.

So, they are both linear combinations of the functions e to the z and e to the minus z, both of these functions e to the z and e to the minus z are analytic everywhere in the finite complex plane. So, their entire functions. So, the linear combination of an entire function is also entire.

So, immediately we see that, the hyperbolic cosine function and the hyperbolic sine function both of these are also entire functions.

So, the derivatives are also readily written down. So, the derivative of e to the z is e to the z, the derivative of e to the minus z is minus e to the minus z; combining this we can immediately obtain the result that the derivative of the cosine function. The hyperbolic cosine function of z is the same as the hyperbolic sin of z and the derivative of the hyperbolic sine function is the hyperbolic cosine function.

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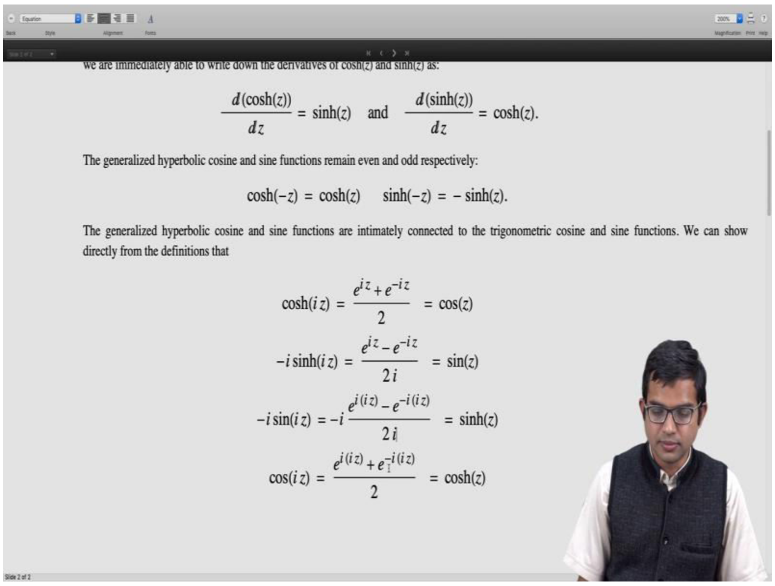

So, the generalized hyperbolic cosine and sine functions also remain even and odd respectively, right. So, the hyperbolic cosine of minus z is equal to the hyperbolic cosine of z as can be directly seen just by changing the sign in the original definition. And the hyperbolic sine function of minus z is minus hyperbolic sin of z, so it is an odd function. So, the hyperbolic functions are intimately connected to their trigonometric counterparts.

So, the cosine and sine functions and the hyperbolic cosine and sine functions are very closely related. So, we can show this directly from the definition. So, cosine of i z so, instead of taking the hyperbolic cosine of z, if you take the hyperbolic cosine of i z; so then we just plug it into the definition we see that, it is i e to the i z plus e to the minus i z the whole thing divided by 2, which is nothing, but the cosine of z.

So, we have this result that the cosine of i z is nothing, but cosh of z. And likewise we can show that minus i times the hyperbolic sin of i times z is e to the i z minus e to the minus i z divided by 2 i, right. So, this comes from plugging in here in this equation and then using you know here use this minus i to bring this i down here. So, then we see that this is nothing, but the definition for the sin of z. So, which is sin of z.

We can also show that minus i times sin of i z is minus i times, you know just plugging in the definition for sin of i z, which is e to the i times i z minus e to the minus i times i z divided by 2 i. But these i squares will give you minus and so, then you have e to the plus z and e to the minus z the whole thing divided by you know 2.

There is a cancellation of this i and it takes care of all the signs correctly and then you will see that you get exactly hyperbolic sin of z. And once again it is straightforward to verify that, cos of i times z is hyperbolic cosine of z, which can also be seen from this first relation, right.

So, if the hyperbolic cosine of i z is cos z, if you put in place of z, if you put i z; you will get cosh of i z is i square cosine of i square z, but then you know there is this evenness property of hyperbolic cosine. So, immediately you can verify that, cos of i z is the same as the hyperbolic cosh of z.

So, in some sense you would see that from these relations, you know cosine of x has a hyperbolic cosine and sin of x along the real axis will behave somewhat like you know hyperbolic cosine and hyperbolic sine's do along the imaginary axis.

So, in some sense and vice versa, right. So, the manner in which hyperbolic sine's and cosines behave along the real axis is going to be similar to how you know trigonometric functions behave along the imaginary axis. So, we look at you know this aspect a little more carefully ahead.

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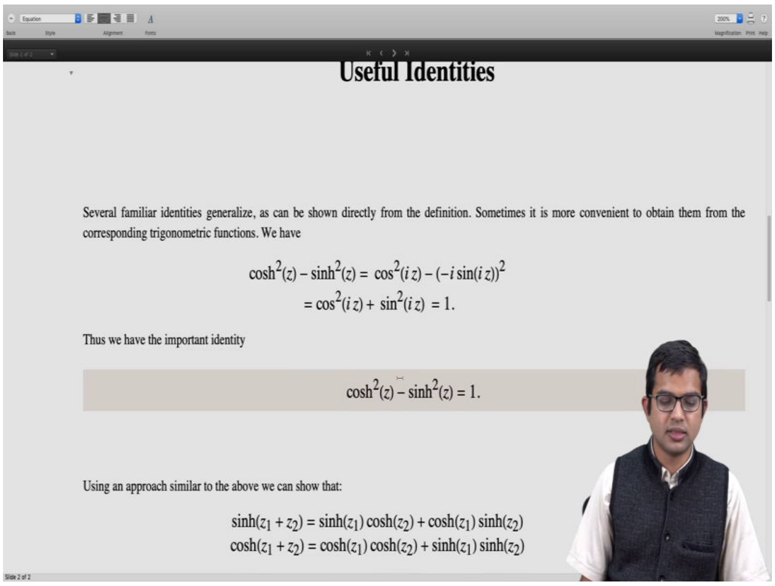

So, let us first look at some identities, one is you know this generalization of you know standard identity of the cosine squared and sine squared. So, if you compute cosine squared of z minus hyperbolic sin squared of z and then we make use of you know these relations of cosine of z and hyperbolic sin of z; so we have cosh squared of i z minus minus i times sinh of i z the whole squared.

So, I am using this identity here. And so, then minus i squared becomes you know minus 1 and so, this becomes and in in place of. So, I have cos squared of i z plus sin squared of i z and we have seen that you know this must be equal to 1, no matter what z is for; this whole, this is a generalized result it holds true for any complex number.

And therefore, we have the result that hyperbolic cosine squared of a complex number minus hyperbolic sin squared of the sine complex number must be equal 1, right. So, in general this is a useful trick, if you want to derive any identities pertaining to hyperbolic cosine and hyperbolic sine functions, just convert into the corresponding sine and cosine function using these identities and then you can use you know their properties to work this out, right.

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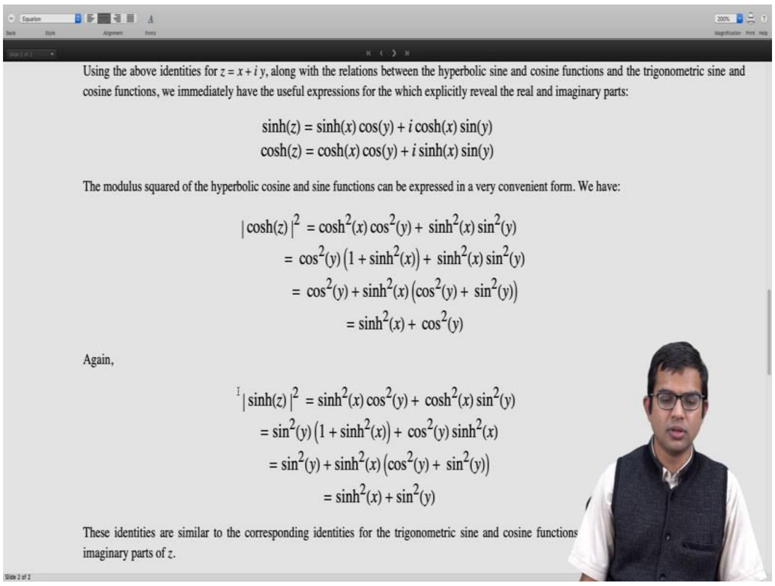

So, we can show that hyperbolic sinh of z 1 plus z 2 is equal to sinh hyperbolic of z 1 times cosh hyperbolic of z 2 plus cosh hyperbolic of z 1 times cosh sinh hyperbolic of z 2, ok.

So, using an approach similar to the above or directly from first principles using the definition of hyperbolic sinh and hyperbolic cosh, we can show that the hyperbolic sinh of z 1 plus z 2 is the same as sinh hyperbolic of z 1 times cosh hyperbolic of z 2 plus cosh hyperbolic of function of z 1 times sinh hyperbolic function of z 2 and a similar identity involving the hyperbolic cosine of z 1 plus z 2, right.

So, this is something which follows directly from the definition right, which you can check explicitly. Now, if you use this these relations and in place of z 1 and z 2, we just put z is equal to x plus i y; then we have sinh of x plus i y which is just z is equal to sinh hyperbolic of x times cosh hyperbolic of i times y, but which is the same as cos of y.

And then we have cosh hyperbolic of x times sinh hyperbolic function of i times y, which brings out this i and then you have sin of y. And likewise we can show that, cosine hyperbolic of z is cosh hyperbolic of x times cos y plus i times sinh hyperbolic of x times sin of y.

Now, but these are actually nothing, but you know expressions for which spell out the real part and imaginary part of these complex numbers, sinh hyperbolic of z and cosine hyperbolic of z, right. So, these are useful identities and it also allows us to work out the modulus squares of these complex numbers. So, if we use a you know a line of argument which is very similar to what we did when we were working with you know sine's and cosines modulus squared, right.

So, in some sense you will see that, it is really the role of x and y get reversed right; when you are working with cosines and when you are working with hyperbolic cosines, eventually you get mod of cosh squared of z is equal to sin hyperbolic squared of x plus cos squared of y, you can check this. And we also have the result, mod of sin hyperbolic of z the whole square is equal to sin hyperbolic square of x plus sin squared y.

So, indeed you know these functions are unbounded, as we are anyway familiar with even for real variables; it is only real variables, which makes it actually unbounded. If you stick to the imaginary you know you know line, if you put x equal to 0; then in fact these functions become very you know they become like the trigonometric functions, like we were saying earlier. And so, in fact they will be bounded and if your complex number that you are considering is purely imaginary.

But so, in general of course, neither of these functions is bounded; but it is useful yeah to have this kind of an identity for the value of the modulus squared.

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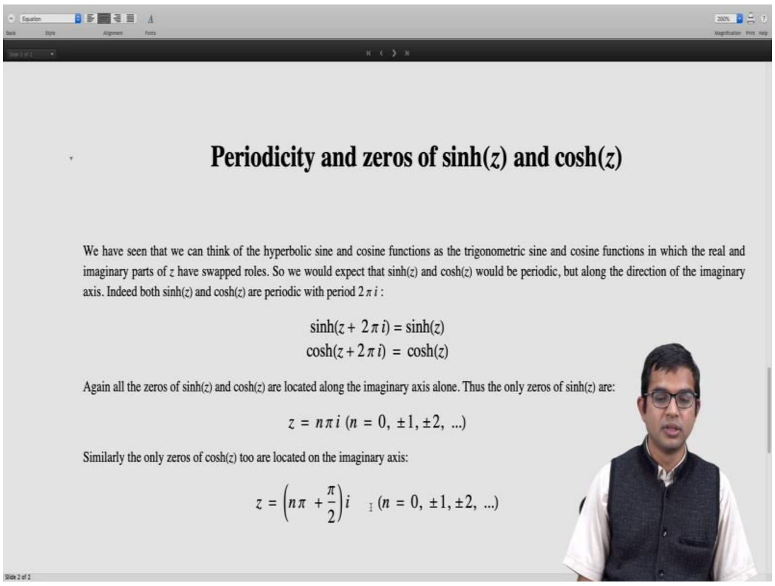

So, these functions are also periodic and their periodicity is, you know it is their period is 2 pi i. So, we have seen how sine hyperbolic we can think of it as really a kind of sinusoidal function, but you know whatever is happening along the x axis is going to happen along the y axis and vice versa. And so, instead of 2 pi, so you have a period which is 2 pi i.

So, sinh hyperbolic of z plus 2 pi i is equal to sinh hyperbolic of z, which is something you can verify directly from first principles from the definition. Again cosh hyperbolic of z plus 2 pi i is equal to cosh hyperbolic function of z. And so, now, the zeros of these functions lie along the imaginary axis along, nowhere else on the plane can you find any other zeros for these two functions.

So, the only zeros are located at z equal to n pi i for sin hyperbolic of z and the only zeros for cosine hyperbolic of z are located at these points z is equal to n pi plus pi by 2 the whole times i again along the imaginary axis. So, once again this is a result which would follow directly from the relationship between sinh hyperbolic of z and sinh of i z and again cosine hyperbolic of z and cosine of i z, right.

So, we can exploit that to work out the zeros of these functions right that is one way, there are other ways as well, ok.

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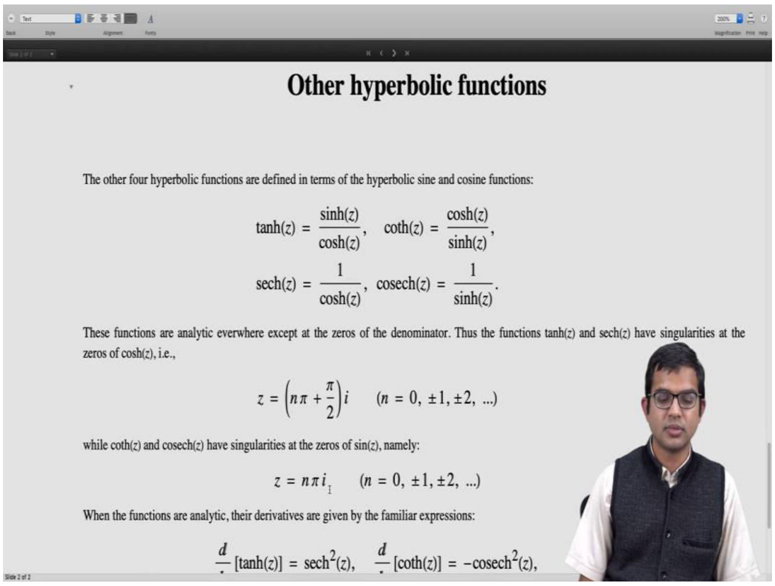

So, we can also define these allied hyperbolic functions, tanh hyperbolic of z is a sinh hyperbolic of z divided by cosine hyperbolic of z, coth hyperbolic of z is cosh of z divided by sinh of z, secant hyperbolic of z is 1 over cosine hyperbolic of z, cosecant hyperbolic of z is 1 over sinh hyperbolic of z.

So, analogous to how you know these kinds of functions worked with trigonometric functions; these functions as well here are all analytic at all points except, where the denominators have zeros. So, whenever cosh hyperbolic of z has a zero; so namely points like here z is equal to n pi plus pi by 2 times i along the imaginary axis, you know these functions tan h of z and secant h of z both of these functions are analytic at every point except at these points.

And again coth hyperbolic and cosecant hyperbolic of z have singularities at these points z is equal to n pi i all along the imaginary axis; everywhere else you know these functions are also analytic.

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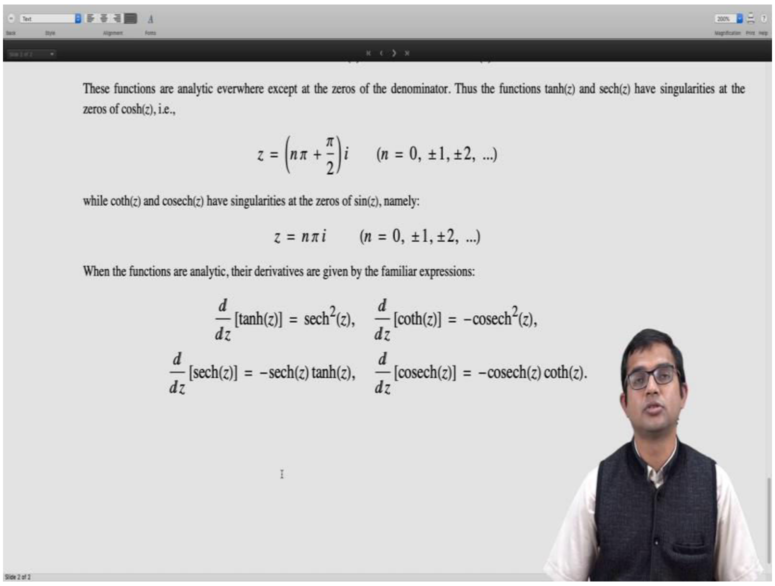

And wherever they are analytic, their derivatives are also readily written. Now, derivative of tan hyperbolic of z is secant hyperbolic squared of z, d by d z of coth hyperbolic of z is minus cosecant hyperbolic squared of z, d by d z secant hyperbolic of z is minus secant hyperbola of z times tan hyperbolic of z, d by d z of cosecant hyperbolic of z is minus cosecant hyperbolic of z times cot hyperbolic of z, right.

All of these expressions are ready generalizations of the analogous result we have seen, when we restrict z to be a real number, ok.

So, that is all for this lecture, we have looked at hyperbolic functions and the generalization to complex variables in this lecture.

Thank you.