Mathematical Methods 2

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Module - 02
Complex Variables
Lecture - 18
Trigonometric functions of complex variables

So, in this lecture, we will look at how to generalize Trigonometric functions. So, this will rely on our generalization of the exponential function, and we will also see how many of the properties that we are familiar with for trigonometric functions of a real variable many of them will also generalize to the complex scenario. And we will also look at how some properties are new and which come about because you are looking at the function of a complex variable.

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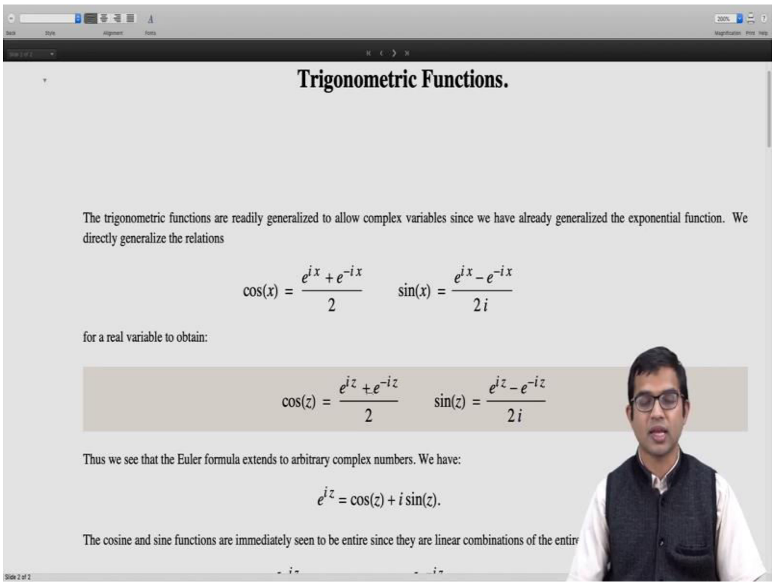

So, the starting point is the recollection that we can think of cosine of x and sine of x, when x is a real variable as e to the i x plus e to the minus i x divided by 2 and e to the i x minus e to the minus x divided by 2 i right. So, which is really already using the fact that we have generalized the complex function to include complex variables. And it is really a rephrasing of the Euler formula in some sense right.

But now we will elevate x to the status of a complex number. And use this as a way to generalize the idea of the cosine of a complex number. So, we define the cosine of a complex number z as e to the, i z plus e to the minus i z divided by 2, and the sine of z to be e to the i z minus e to the minus i z divided by 2 i right. So, in fact, this can be thought of as a generalization of the Euler formula to arbitrary complex numbers. So, we have e to the i z is equal to cos z plus i times sin of z right.

So, we can immediately see that from this definition both cosine of z and sine of z are entire functions right, because e to the i z and e to the minus i z are entire functions. We have seen that the function e to the z or e to the minus z both of these are analytic everywhere in the complex plane in the finite complex plane.

Therefore, they are entire, so if you take any linear combination of entire functions you are going to get another entire function. So, the cosine of z is an entire function analytic everywhere in the finite complex plane and so is the sine of z.

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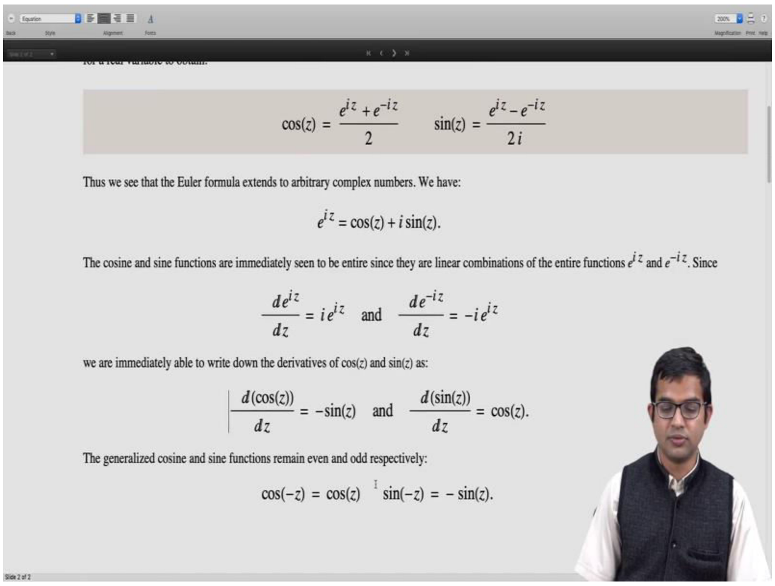

So, if you take the derivative of the exponential function e to the i z, you get i times e to the i z and if you take the derivative of e to the minus i z you get minus i times e to the i z. So, it is analytic everywhere and the derivatives are explicitly written down.

So, using this we can immediately write down the derivatives of cosine of z and sine of z. So, d by d z of cosine of z you can quickly convince yourself. It is actually nothing but minus sine of z and d by d z of sine of z is nothing but cosine of z according to this generalized definition, so these two relations are the familiar relations which we know and which we have used from high school days.

And so that these relations that we are familiar with pertain to real variables, but they also extend to complex variables. Also these generalized cosine and sine functions retain their even and odd character respectively. So, cosine of minus z, you can verify from first principles from the definition, is the same as cosine of z, and sine of minus z is minus sine of z right.

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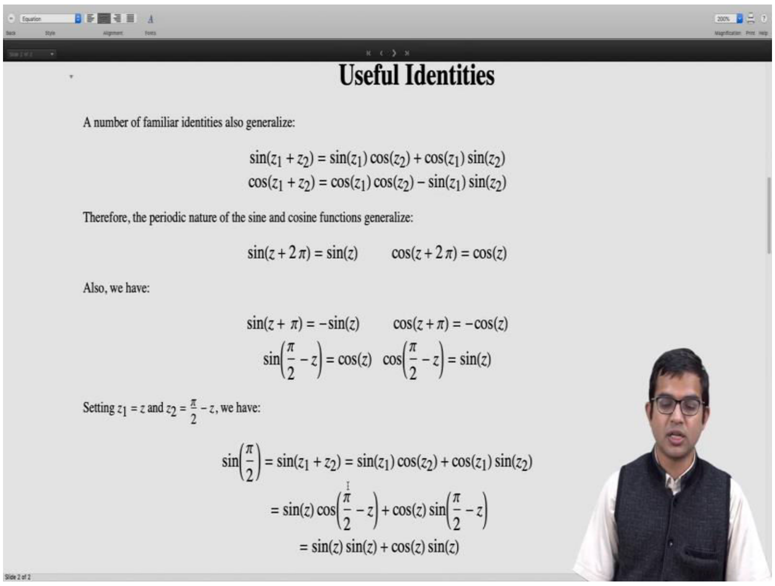

So, there are several useful identities which also generalize, and all of this goes back to the definition of cosine of z and sine of z. You can directly derive all of this right. So, for example, sine of z 1 plus z 2 is equal to sine of z 1 times cosine of z 2 plus cosine of z 1 times sine of z 2 and cosine of z 1 plus z 2 is equal to cosine of z 1 times cosine of z 2 minus sine of z 1 times sine of z 2 right.

So, this is something that can be verified by just directly plugging in this expression for cosine and for sine, these two expressions. And then working out the algebra, one can show that indeed these identities carry through. So, the periodic nature of sine and cosine functions also generalize so which you can actually directly see from plugging in here this these expressions sine of z plus 2 pi is the same as sine of z and cosine of z plus 2 pi is the same as cosine of z, no matter what complex number z you are looking at.

But we also have these relations sine of z plus pi is minus sine of z and cosine of z plus pi is minus cosine of z. And also we have the relation sine of pi by 2 minus z is equal to cosine of z, this also generalizes cosine of pi by 2 minus z is equal to sine of z right. So, these identities can also be worked out using these identities right. So, this is something that you can check for yourself that these are all direct consequences ultimately of the definition of sine of z and cosine of z right.

So, these are familiar identities, but what is important to emphasize is that they carry through exactly the same form even when we allow z to be a complex number. Now, if we set z 1 to be z and z 2 to be pi by 2 minus z, so if we add these two of course, it is just pi by 2.

So, sine of z 1 plus z 2 which is pi by 2 sine of pi by 2 is sine of z 1 plus z 2 which is the same as sine z 1 cos z 2 plus cosine z 1 sine z 2 which is sine of z times cosine of pi by 2 minus z plus cosine of z times sine of pi by 2 minus z, but cosine of pi by 2 minus is sine of z, and sine of pi by 2 minus z is cosine of so this should be cosine of z, sine of pi by 2 minus z is cosine of z.

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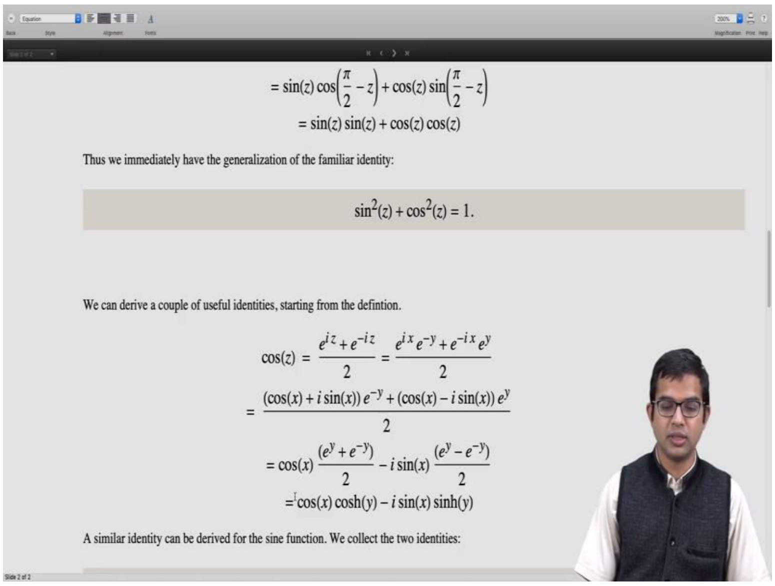

And from which we immediately get the identity sine square z plus cosine square of z is equal to 1 right. So, this is the generalization of the familiar identity which we have used you know a lot. So, there is a couple of useful identities which we can you know derive again starting from the definition. So, cosine of z is just e to the i z plus e to the minus i z divided by 2. So, you can write this as so write z as x plus i y. So, you get e to the i x times e to the minus y plus e to the minus i x times e to the y.

Then if we expand so we get cos x plus i sin x times e to the minus y plus cos x minus i sin x times e to the y whole thing divided by 2. And then you collect terms carefully. So, you have cosine of x times e to the y plus e to the minus y divided by 2 minus i times sine x times e to the y minus e to the minus y divided by 2.

But then we immediately see that e to the y plus e to the minus y by 2 is nothing but cosh the hyperbolic cosine of y and e to the y minus e to the minus y divided by 2 is hyperbolic sine of y. So, we get this identity cosine of z is cos x hyperbolic cosine of y minus i times sine x sine hyperbolic of y right.

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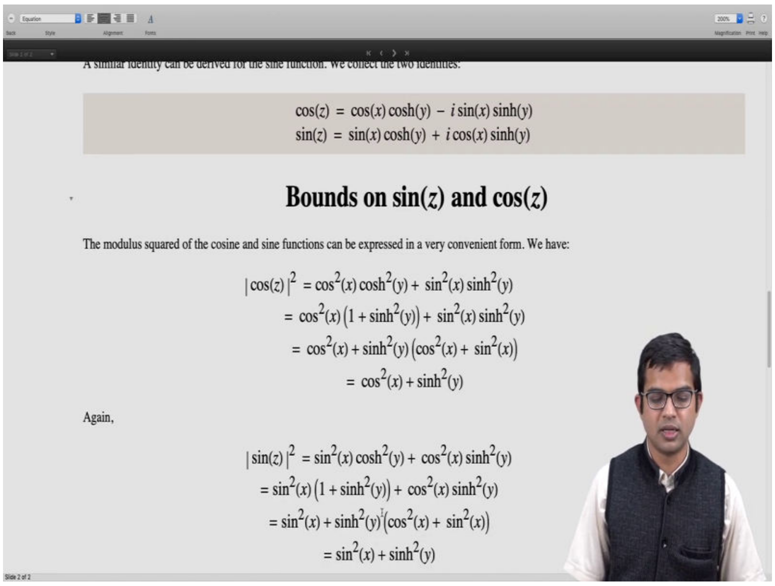

We can derive a similar identity for the sine function which we you know state and allow you to verify yourself sine of z is sine x hyperbolic cosine of y plus i times cos x hyperbolic sin of y. So, these identities right, so you can also think of these as you know an explicit expansion of these complex numbers after all cos z is a complex number sin z is the complex number and so you can think of these as expressions for the real part and the imaginary part right.

And this, these expressions will also allow us to investigate bounds on sine of z and cosine of z right. So, when we are working with the trigonometric functions of a real variable, so we know that both sine and cosine functions give you numbers which necessarily must lie between minus 1 and plus 1.

They can never get out of this bounded region. But on the other hand, when we are working with complex numbers we see that in fact they are unbounded from above right. So, mod of cosine of z square you know is just the real part squared plus the imaginary part squared which you expand, and then you collect terms, and then you use the identity that cosine squared of y is 1 plus hyperbolic sine squared of y, so plus sine squared of x times hyperbolic sine squared of y, then you collect terms again.

So, you know cos squared x comes out then hyperbolic sin squared y times cos squared x plus sin squared of x which is just 1. So, you get cos squared of x plus hyperbolic sin squared of y.

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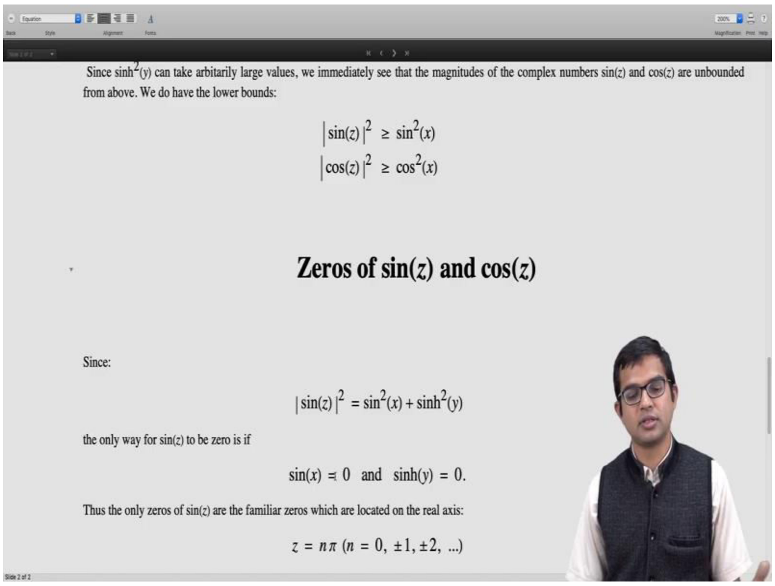

And similarly you can show with a similar type of algebra that mod of sine z squared is sine squared x plus hyperbolic sine squared of y. So, we see that since the hyperbolic sine of y can take arbitrarily large values. If y is non-zero meaning it is a complex number, it is a genuine complex number, then indeed the value the modulus of cosine of z or and the modulus of sine of z can be arbitrarily large right. So, they are unbounded.

But we can also write down these lower bounds after all mod of cosine of z squared and mod of sine of z squared you know is seen to be the sum of two squares, therefore, each of the mod of cosines you know z squared must be greater than or equal to sine squared of x and mod of cosine z sine.

So, mod of sine z squared is greater than or equal to sine squared of x, and mod of cosine squared is greater than or equal to cos squared x. Sometimes, these inequalities you know are useful to put these kinds of lower bounds right. So, this what about the zeros of sine of z and cosine of z. We know for sure that you know this function is periodic, and it keeps hitting zero as you go along the real axis.

So, it turns out that there are no other zeros. All the zeros of these functions are located only on the real axis right. So, one way to see this is mod sine of z squared is you know the sum of the these two squares if sine of z is 0, the only way that can happen is the modulus itself is 0 and which can happen only if each of these positive numbers which add up to form this number are separately 0.

So, sine of x must be 0, and sine of hyperbolic cosine of y is 0. And which immediately means that y must be 0 and sine x is 0 tells you that it can happen only when x is equal to n pi, where n is some integer right. So, these are the familiar zeros of the sine function. So, by generalizing the sine function to allow for complex variables, we do not find any new zeros, the zeros are still those on the real axis.

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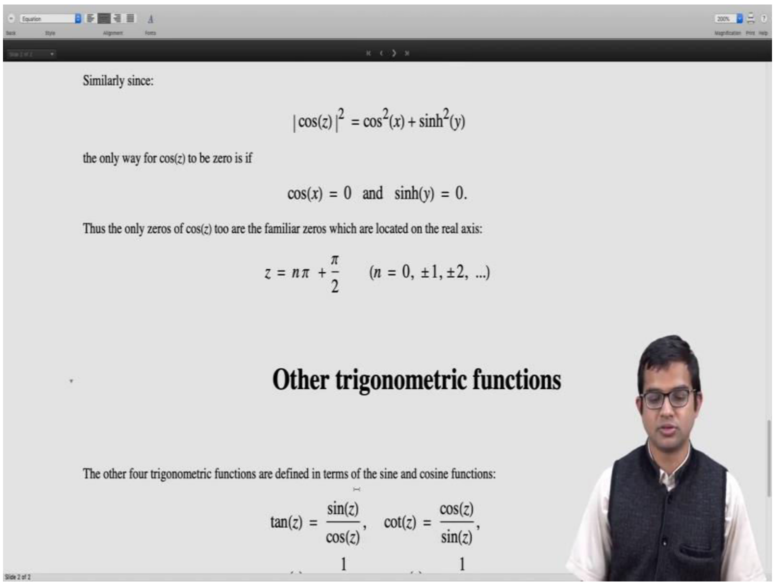

And likewise mod of cos z squared if this gone be 0, then cos squared of x must be 0, and sin hyperbolic square of y is 0. And therefore, y must be 0, and cos x is 0 implies you again get these familiar zeros. So, z is equal to n pi plus pi by 2 where n is some an arbitrary integer. So, using these definitions for the sine and the cosine, we can also find you know we can also define other allied trigonometric functions.

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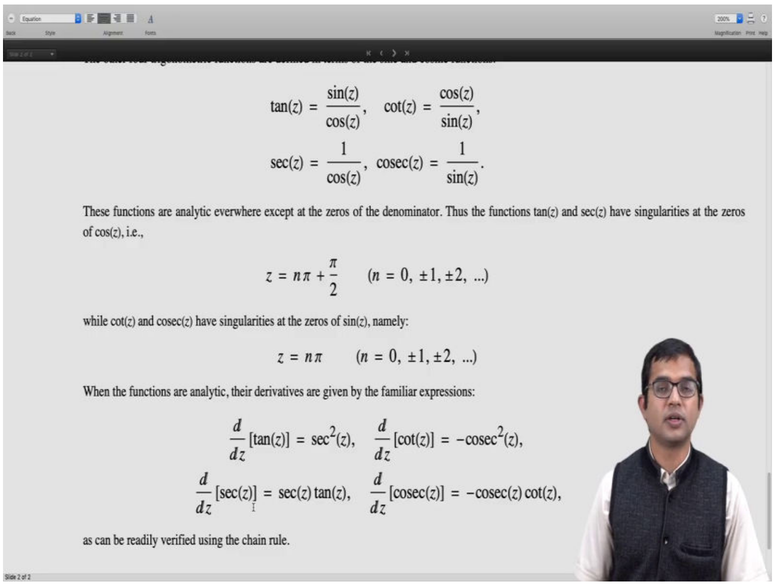

So, we can define tan of z to be sine of z divided by cosine of z, cot of z to be cosine of z divided by sine of z, secant of z to be 1 over cosine of z, and cosecant of z to be 1 over sine of z. Now, these functions are also analytic everywhere except those points where the denominator goes to 0 right. So, tan of z is analytic unless cosine of z is 0, and which we have seen happens at these points z equal to n pi plus pi by 2. They all lie on the real axis.

So, these are the singularities of these functions tan of z and secant of z both of these you know analytic everywhere except at these point z is equal to n pi plus pi by 2, where n is some integer. And again cot of z and cosecant of z you know these both of these functions are analytic everywhere in the complex plane except at these points on the real axis, where sine of z has the 0, you know these are the points z is equal to n pi where n is a an integer.

So, when these functions are analytic that their derivatives exist and in fact they have this, these same familiar expressions which we are used to from considering these functions of a real variable.

So, derivative of tan z is going to be secant squared of z you know according to the way we have defined it, derivative of cot of z is minus cosecant squared of z, derivative of secant of z is secant z times z, derivative of cosecant of z is equal to minus cosecant of z cot c, all of these identities just carry through directly at all points where these functions are analytic ok. So, that is all for this lecture.

Thank you.