Mathematical Methods 2

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Module - 02
Complex Variables
Lecture - 15
The Exponential Function

So, we have looked at the notion of analyticity, we have been looking at functions of a complex variable. We have seen some properties that analyticity endows a function with. There are many more properties which will also be discussed as we go along, but analyticity is a nice thing to have right.

So, it allows us to make general statements about functions; functions which are analytic have certain nice properties in the whole region of analyticity. And therefore, it is a desirable thing to endow our function with.

So, in this lecture, we will look at how the exponential function which is something which was defined for real variables to start off, and how the exponential function can be generalized for complex variables, keeping in mind the requirement or the desirability of making it an analytic generalization right.

So, we will look at how to start with the exponential function for a real variable and generalize it so that it remains analytic in as large a region as possible. We will also look at a similar exercise in the context of many other functions as we go along, but in this lecture it is all about the exponential function ok.

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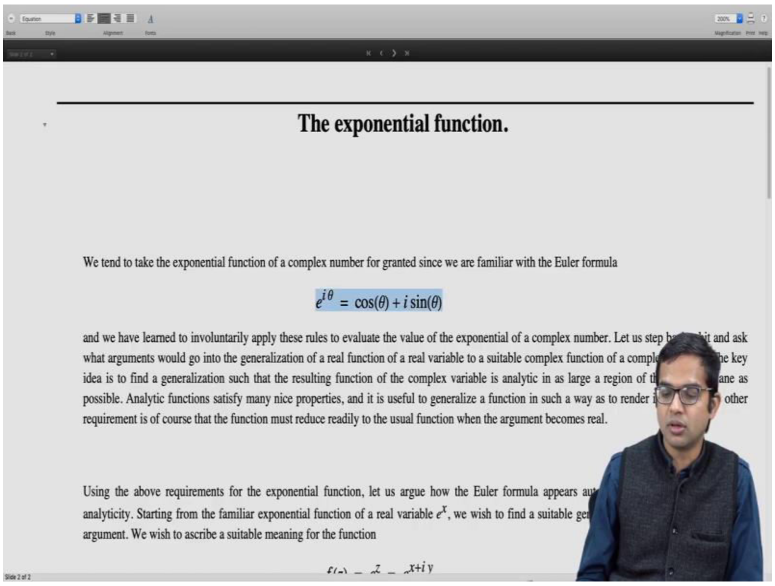

So, we are all familiar with the Euler formula. So, it says that e to the i theta is equal to cos theta plus i sin theta. And one argument to see how this comes about is to write down this Taylor expansion on the left hand side. So, e to the i theta actually expanded, and so you blindly take the argument to be i theta.

And then you collect all the real terms at the imaginary term separately, and you can sort of argue that the series corresponding to cosine theta will be the real part, and the series corresponding to sin theta will be the imaginary part right. So, this is one way to come up with a justification for this formula right.

So, in this lecture, we will look at how starting from the idea of an exponential of a real variable, if you wish to generalize this function to a to apply it on complex numbers, how can we do this in such a way that we have analyticity for this function in as larger region as possible right.

So, in general when we are generalizing functions so called to include complex variables, you know there are two requirements. One is to ensure that the function reduces to the same function that we are used to when instead of a complex number, you put a real variable. So, if you have z equal to x plus i y and if you put y equal to 0, it must give back the original function right.

So, this is an absolute minimum requirement; otherwise you would not even call it a (Refer Time: 03:34) generalization, but also we also would require; we would require that the generalized function is analytic in as large a region as possible. So, let us see how if we try to do this with the exponential function. We can actually sort of get another alternate way of justification for this Euler formula ok.

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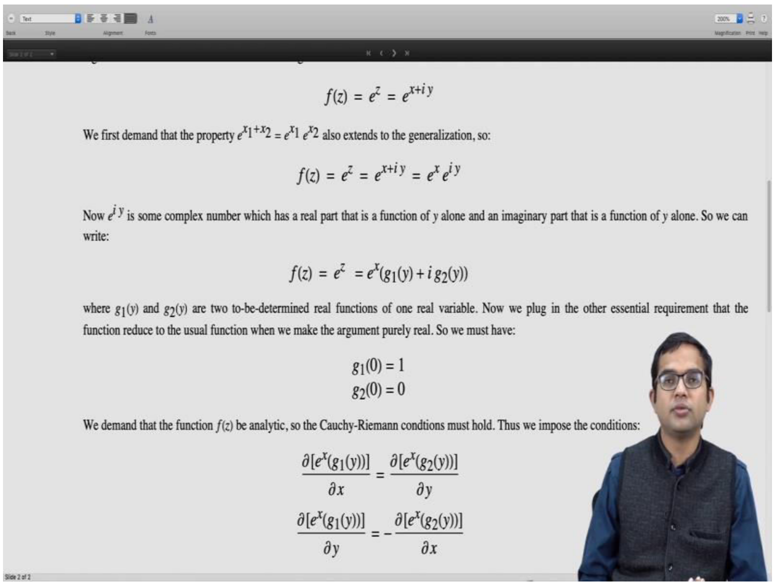

So, we start with the function e to the x where x is a real variable. And then we ask how can we give meaning to the function f of z is equal to e to the z where z is x plus i y right. So, we start by demanding that this function-2 should have you know this familiar property e to the x 1 plus x 2 is equal to e to the x 1 times e to the x 2.

So, we are going to give meaning to this e to the z, but we also expect that this function will also satisfy this kind of a property. So, we should be able to write e to the x plus i y as e to the x times e to the i y right. So, e to the x is the same e to the x that we already know is just exponential of a real variable. Now, e to the i y is where we have to provide meaning for this quantity e to the i y.

So, it has a real part and an imaginary part. So, we can you know think of this real part as some function g 1 of y right. So, it is necessarily a function only of y because the input is only y. And then it has an imaginary part which is g 2 of y right. So, as far as this function f of z is concerned, this function also has a real part and an imaginary part right, so which we are used to thinking of as u of x comma y, and v of x comma y – these are the real parts and imaginary parts.

But here this u of x comma y, and v of x comma y have this special form. So, it should be u of x comma y must be of the form e to the x times g 1 of y, and v of x comma y is of the form e to the x times g 2 of y. Now, these functions g 1 of y, and g 2 of y must also satisfy this property that when you put y equal to 0, g 1 of 0 must go to 1, and g 2 of 0 must go to 0 right.

Only if this happens would we get back e to the x when y is 0 right. So, this is also an essential requirement to get a valid generalization of the exponential. So, now, we want this resulting function to be analytic in as large a region as possible. So, if this function is analytic, then it must satisfy Cauchy-Reimann conditions.

So, we have u of x comma y is e to the x times g 1 of y, and v of x comma y is e to the x times g 2 y. And if you plug in the Cauchy-Reimann conditions with you know for these two functions u and v. We must impose the conditions dou by dou x of e to the x times g 1 of y must be equal to dou by dou y of e to the x times g 2 of y. And also dou by dou y of e to the x times g 1 of y must be equal to minus dou by dou x of e to the x times g 2 of y right.

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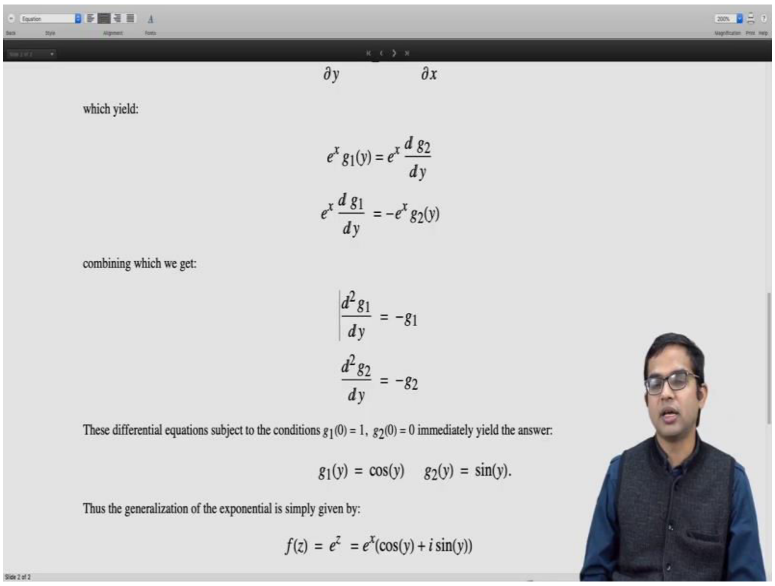

So, if we carry out these partial derivatives, this is an x differentiation with respect to x. So, it is just e to the x. And so you get the left hand side is e to the x times 0 to y, and the right hand side e to the x is for all practical purposes is just a constant as far as this partial derivative is concerned. So, you can pull it out. And then dou by dou y will become d by dy because g 2 is your pure function of y.

So, you have e to the x times d g 2 by dy. And then the second equation becomes e to the x times g 1 d g 1 by d y must be equal to minus e to the x times g 2 of y right. So, if we combine these two conditions, so e to the x will cancel throughout. So, g 1 of y is equal to d g 2 by dy, but g 2 of y is minus d g 1 by dy.

So, if you take the derivative, you get d square g 1 by dy squared is equal to minus g 1. And likewise if you take the derivative of you know with respect to y in the first equation, then you get d squared g 2 by dy is equal to d g 1 by dy, but g d g 1 by dy is the same as minus g 2.

So, indeed both of these have the same second order equations which we know how to solve right. So, this is nothing but the simple harmonic oscillator problem. So, the solutions are either cos y or sin y or some arbitrary linear combination except in this case it is not quite arbitrary because we are given that g 1 of 0 must be equal to 1 and g 2 of 0 must be equal to 0 right.

So, that comes from this requirement that this function generalization will reduce to just exponential x when we put y equal to 0. So, if we do this, so we can write down immediately that g 1 of y must be equal to cos y and g 2 of y is equal to sin y. Therefore, a natural generalization for the exponential as we go from you know exponential of real number to exponential of a complex number is to say e to the z is eta e to the x times cosine of y plus i times sin of y.

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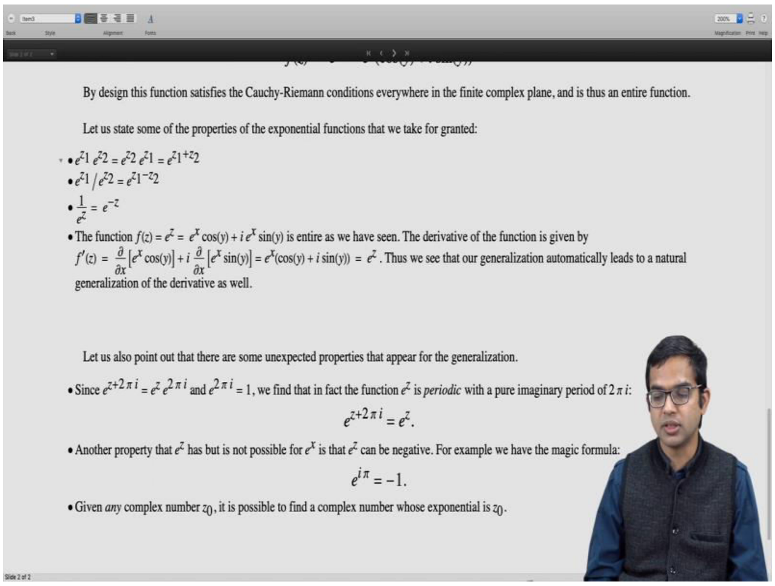

Now, if you put x equal to 0 of course, this will immediately give us the Euler formula basically. So, it is the statement that e to the i, i theta or i y in this case. So, in place of e to the z you have e to the i y this is cos y plus i sin y. So, this is a way to argue for the Euler formula right.

So, it is a generalization of the exponential function such that the generalization is analytic. In fact, it turns out that this function e to the z is analytic in the entire finite complex plane. So, there are no singularities anywhere in the finite complex frame. So, it is an entire function.

So, we have achieved both our goals and to the extent which is possible to the maximum extent possible. So, we have got an entire function. So, it is very nice. And so there are some properties of exponential functions that we take for granted. But we will state it here you know which well I mean it can be verified using these first principles right.

So, this definition, so if you take e to the z 1 times e to the z 2, it is the same as whether you do e to the z 2 times e to the z 1 and that is the same as e to the z 1 plus z 2. Again e to the z 1 divided by e to the z 2 is e to the z 1 minus z 2, and 1 over e to the z is e to the minus z right. So, these all follow from this definition of the exponential of a complex number.

And then we can also take the derivative of this function. It is an analytic function, everywhere it is an entire function. So, the derivative is well-defined everywhere in the finite complex plane. And so if you take the derivative, we have seen that the value of the derivative is just given by dou u by dou x plus i times dou v by dou y.

So, in this case, it is just going to turn out to be e to the x times cos y plus i times you know e to the x times sin y which you can pull out the e to the x, and then you can rewrite cos y plus i sin y. And then you just identify that this function is nothing but e to the z right.

So, if you take the derivative of the exponential function with respect to z right it is I mean it is evidently not dependent on z star. So, it is analytic everywhere. And so the derivative is well-defined and its value is the same as you know the value you have given to the function itself, which is very nice.

So, we see that not only does the function get a valid generalization, but its derivative formula also looks the same as if we are working with a function of a real variable right. So, these are all properties which are familiar from you know properties of exponential of a real variable, but there are also some properties of the generalized function, exponential of a complex number which are a bit counter intuitive.

So, let's look at some of these. One is that it turns out that this exponential of z is actually a periodic function right. So, we do not think of the exponential as a periodic function. But, when you generalize it, so you have e to the z plus 2 pi i is the same as e to the z times e to the 2 pi i, and e to the 2 pi i is 1.

So, we find that in fact e to the z is periodic with a pure imaginary period of 2 pi i. So, you can add any integral multiple of 2 pi i to your z and if you take the exponential of this number, you get back the same answer as for the original complex number itself. So, in fact, the exponential function is a periodic function in the complex plane.

So, another property of e to the z that is not there for e to the x is that e to the z can also be negative right. So, in general, it can be a complex number, and in fact, it can also be negative right. So, this gives the magic formula e to the i mean this is seen in the magic formula e to the i pi is equal to minus 1 right so which is the famous formula which connects all the fundamental mathematical constants.

Also in fact you know you can find a complex number such that e to the such a complex number is whatever complex number you want, not just a negative one. But, in fact given any complex number z naught it is possible to find a complex number whose exponential is z0 right.

So, you know this is a statement we will just make here. But this is the seed of how to find such a complex number the exponential of which will be a given complex number.

If we go further in that direction that will lead us to the next function which we will discuss in detail in a separate lecture, namely the logarithmic function. But, as far as this lecture is concerned, the focus is on the exponential function and that is all for this lecture.

Thank you.