Mathematical Methods 2
Prof. Auditya Sharma
Department of Physics
Indian Institute of Science Education and Research, Bhopal

Module - 02
Complex Variables
Lecture - 13
Analytic functions

So, we have looked at the notion of differentiability for a function of a complex variable and we have seen how it is quite restrictive. So, it automatically implies constraints for the real part and the imaginary part in terms of Cauchy-Riemann conditions.

So, it turns out that we want to constrain it a little more. So, differentiability of course, is very good. It gives you some nice properties for your overall function f of z, but we want to do slightly more than just differentiability and so, that is where we introduce the notion of analyticity.

So, we started with continuity and then when we went from continuity to differentiability already we saw that there is quite a lot of restriction that comes about. Analyticity allows us many very interesting properties for functions - these functions are slightly more than just differentiable, right.

So, we discuss what analyticity is in this lecture and also look at several examples and some properties of analytic functions.

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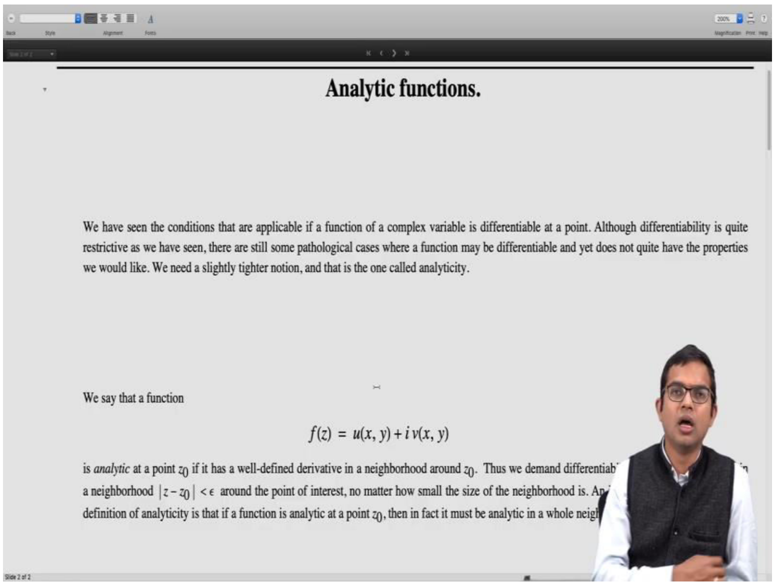

So, we say that a function is analytic if it is differentiable not at just one point, but if it is differentiable in a whole neighbourhood right around this point z0. So, you have a function: it has a real part u and an imaginary part v both of them are functions of x and y and so,

And, so, we demand differentiability not at just a point, but in a whole neighbourhood and a neighbourhood is simply some circular region around z0. It does not have to be circular, but for simplicity we can always think of a regiona mod of z minus z0 less than epsilon around the point of interest. And, so, no matter how small this size could be, it has to be some positive epsilon, positive real number epsilon.

And, so, an immediate consequence of this is a function is never analytic just at a point, it is always analytic in a whole region around it, right. It is because if a function is differentiable at a point and if it is differentiable in a region around this point. So, if you take any other point in that region you can always argue that the function is differentiable in a delta neighbourhood on that point as well. So, this is clear.

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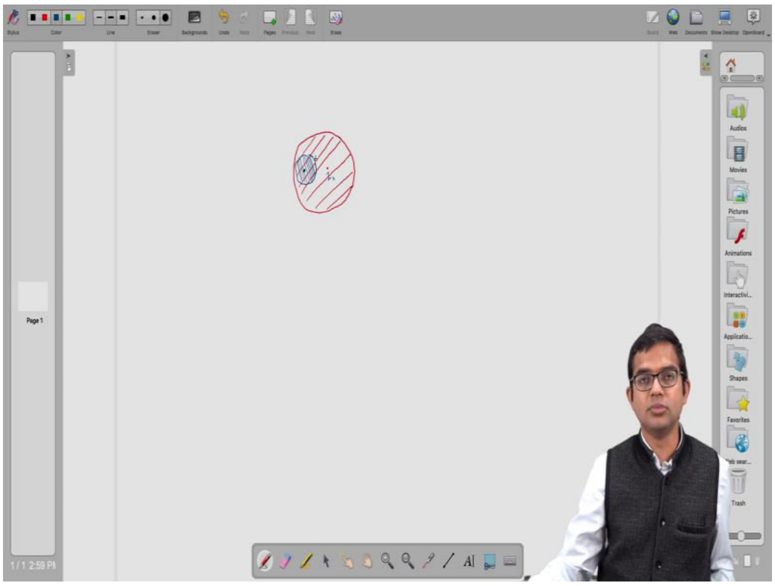

If you just draw a little picture, you have a point z0 and then you consider an epsilon neighbourhood. I have drawn a rather big region. I have exaggerated it for ease of understanding. So, we say that in this entire region the function is differentiable.

So, you have a z0. So, it means that this function is differentiable in this entire region. So, if you go to any other point, right, so, let me use a different colour. So, if I go to some other point then for sure if I take a small delta neighbourhood around here. So, clearly we will always be able to find a neighbourhood around this point where this function is differentiable at all these points.

So, in fact, analyticity at this point, the red dot z0 implies analyticity at this point which we may call z0 prime. So, analyticity always comes in a whole region. So, an immediate consequence of course, is that analyticity happens in a whole region and then we will look at several properties of analytic functions starting from this lecture and then we will develop this theory further as we go along.

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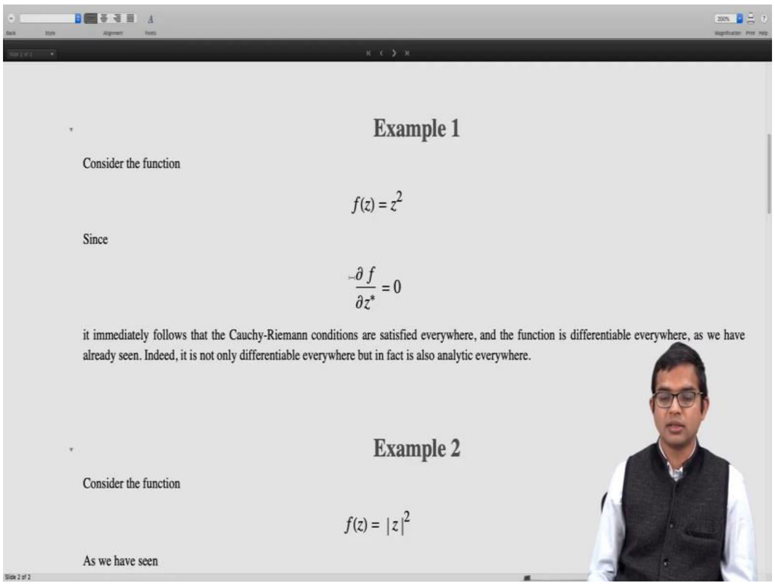

But, let us first look at a few examples. So, we will start with a very familiar example. So, a function f of z is equal to z square. So, since dou f by dou z star is equal to 0. This means immediately that the Cauchy-Riemann conditions are satisfied and this function is differentiable everywhere.

But, now in the backdrop of the current definition right we have just defined what analyticity is. So, we realize that in fact, this function is not just differentiable everywhere, but in fact; it is also analytic everywhere, right. So, it is a very simple example.

Now, let us look at another example which we have also seen multiple times and so, in some sense that is where the need for the definition of an analytic function itself arises. So, let us look at the next example which is also something we have encountered f of z is equal to mod of z square, right.

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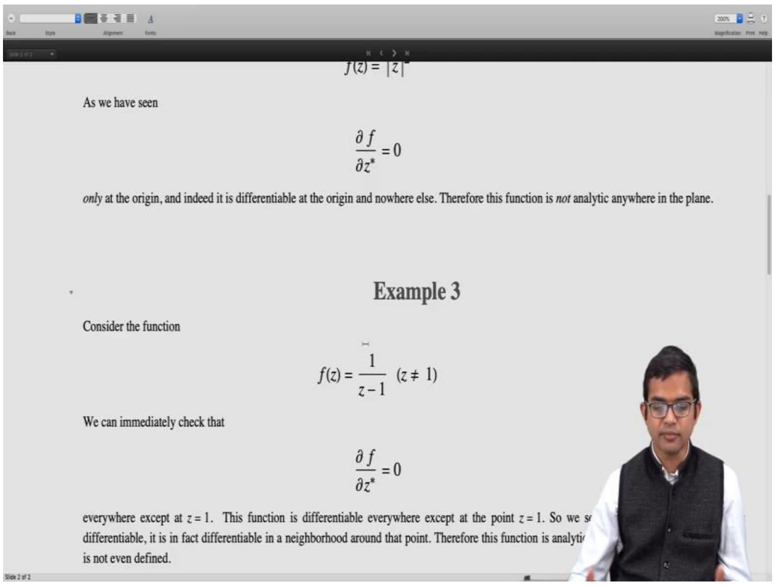

Now, if we try to take the partial derivative of this function f with respect to z star, we have seen that the only way for this to be 0 is if you are at the origin - only at the point z equal to 0 do you have dou f by dou z star is equal to 0. So, everywhere else Cauchy–Riemann conditions do not hold and for sure this function is not differentiable anywhere other than the origin. We have checked that it is differentiable at the origin, but nowhere else.

So, this function is not analytic anywhere in the plane, right. It is differentiable at precisely one point, but that is not enough for it to be analytic at that point; for it to be analytic it must be differentiable in a whole neighbourhood around that point and that is not the case here. So, mere differentiability is not enough for analyticity. So, this function is not analytic anywhere in the plane.

So, in some sense this is and you know examples of this kind are the reason why we introduce the notion of analyticity as something slightly different from the idea of differentiability. So, analytic functions come with many nice properties.

So, it is differentiability in a whole region. And we can play many kinds of games with it, but indeed there are also lots of useful properties which we will work out as we go along. Let us look at another example.

So, if I have a function like this, f of z is equal to 1 over z minus 1, right; it is not defined at the point z equal to 1, but everywhere else it is just 1 over z minus 1. And, we can immediately check that dou f by dou z star is 0 everywhere and it is also a function which is differentiable everywhere.

So, the Cauchy-Riemann conditions hold as you can explicitly check it, but also you know that the function u of x comma y and v of x comma y are both well behaved functions. So, they have these continuity properties that are part of the sufficiency condition.

So, there is no issue it is differentiable everywhere except at the point z equal to 1 where the function itself is not defined. So, there is no question of its derivative existing at that point. So, this function is not only differentiable everywhere except z equal to 1, but it is also differentiable in a neighbourhood around any point other than z equal 1. So, in fact, this function is analytic everywhere except at z equal to 1.

So, if you are slightly far away from z equal to 1 this function is differentiable and then you can always construct a delta neighbourhood around that point by taking a circle of radius whose length is half of the distance between the point z equal 1 and the point of interest right and so, indeed all points inside that circular region are going to be differentiable.

Therefore, all points inside that circular region are going to be analytic. So, therefore, this function is analytic everywhere except at z equal to 1.

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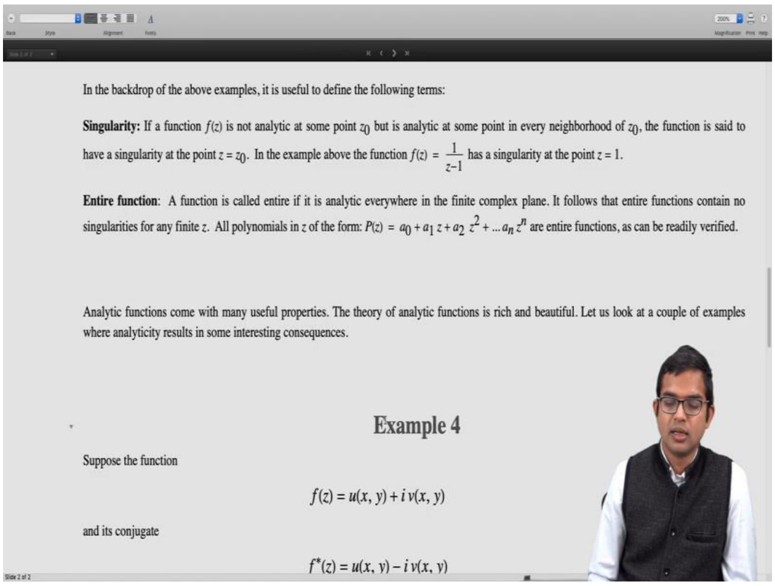

So, in the backdrop of these examples it's useful to define the following terms. So, the idea of a singularity comes straight from the last example which we discussed, right. If your function is analytic everywhere except at this point z equal to 1 in the previous example. So, then we say that z equal to 1 is a singularity, right.

So, formally the definition of a singularity is that if a function has a non analyticity at some point z0, but it should be analytic at some point in any neighbourhood of z0. So, you can consider a neighbourhood no matter how small, but you will be able to find some analytic points in there, then this function has a singularity at that point.

So, we will look at singularities in more detail later on. We will try to exploit the properties of singularities when we work with integrals and so on later on, but also classify singularities. So, there are different kinds of singularities - we will return to this at a later time, but at this point we just want to highlight that a singularity is a point where this function is non-analytic, but inside its neighbourhood there are lots of points which are analytic.

So, the contrast to functions with singularities are what are called entire functions and a function is called entire if it has no non-analyticitics in the entire plane. So, in fact, it is analytic everywhere in the finite complex plane; such functions are called entire functions. There are plenty of examples of entire functions. We will also cover them more extensively as we go along.

But, one natural class of functions which come to mind immediately when we are thinking of entire functions are polynomials of z. So, any function of the form a0 plus a1 z plus a2 z squared plus so on all the way up to an z to n where a0, a 1, a 2 so on all the way up to an are all arbitrary complex constants. Such functions are polynomials in z and they are analytic everywhere - they are entire functions. And, so, we know how to differentiate such functions. We have already looked at the familiar example z squared, but in fact, any arbitrary polynomial also will have a derivative at any finite value of z.

Let us look at some examples where the properties of analytic functions are exploited in interesting ways. So, the theory of analytic functions is vast where many beautiful ideas come together. Let us look at a couple of examples where some of this beauty shines through.

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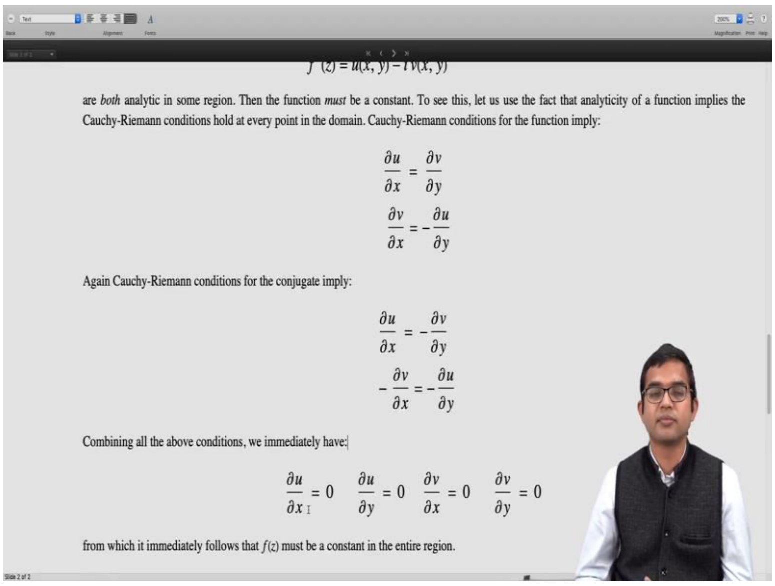

So, suppose we consider some function f of z which has a real part and an imaginary part. Its complex conjugate can be written as f star of z is equal to u of x comma y minus i times v of x comma y. So, suppose we demand that both this function f of z and f star of z must both be analytic in some region, then it turns out that the only way to do that is if the function is a constant function.

There is no non-trivial way in which a function and its complex conjugate are both analytic in a whole region. So, this follows directly from Cauchy-Riemann conditions. Both f of z and f star of z must satisfy Cauchy-Riemann conditions and this turns out to make it so constrained that it becomes a constant. So, the Cauchy-Riemann conditions for the function f immediately imply that dou u by dou x must be equal to dou v by dou y and dou v by dou x is equal to minus dou u by dou y.

Again, the Cauchy-Riemann condition must be satisfied by f star of z, right. So, you have a u and a minus v here. So, dou u by dou x must be equal to minus dou v by dou y and minus dou v by dou x must be equal to minus dou v by dou y and so, if you combine all these four conditions together immediately we see that the only way this can happen is if each of them is 0.

So, if dou u by dou x is equal to dou v by dou y, but dou v by dou y is equal to minus dou u by dou x. So, dou u by dou x is equal to minus dou v by dou x dou u by dou x equal to 0, right. This is the first result and likewise you will see that dou u by dou y is equal to 0 dou v by dou x is equal to 0 and dou v by dou y equal to 0, right.

So, the only way that this can happen is if u of x comma y is a constant and v of x comma y is also a constant, which immediately implies that this function itself is a constant in the entire region. So, analyticity is quite a constraining condition.

So, it is not possible for a function and its complex conjugate to be both you know nice functions if one of them is very nice and analytic and non-trivial, then for sure its complex conjugate is some you know its non-analytic, right, ok in I mean in a whole region, right.

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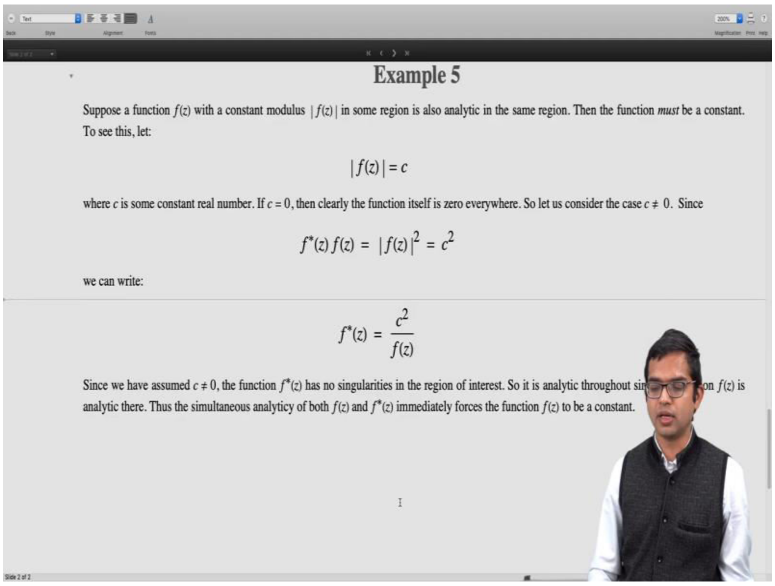

So, let us look at another example where in fact, this property that a function and its complex conjugate cannot both be analytic is exploited to show you to show another property. So, suppose we have a function f of z with a constant modulus mod of f of z.

So, in this entire region it has a constant modulus, but I mean its amplitude could be different. But, it turns out that if this function has to be analytic and also have a constant modulus the only way this can happen is if it must if it is a constant right the whole function itself is a constant, right.

So, to see this let us say that mod of f of z is equal to c, where c is some constant real number, right. It is a, and it is a positive number it is a modulus. Now, if c is 0 then clearly the function itself is 0 everywhere. So, if the modulus of a complex number is 0, then the complex number itself is 0.

So, let us consider the non trivial case which is when c is nonzero if c is 0, then of course, it is constant everywhere and that constant happens to be also 0. But, if c is not 0, then we will use the fact that f star of z times f of z, right. So, the complex conjugate of a complex number of times the complex number is just modulus of the complex number squared which in this case is c. So, this is equal to c squared. So, we can write f star of z as c squared over f of z.

So, now comes the key part of the argument which is that since f of z has a constant magnitude whose value is c, right and we have already said that, that constant value is not 0, right. So, there is no point in this domain where f of z becomes 0 because if f of z were to become 0, then mod of f of z is also 0, but mod of f of z is not 0. Mod of f of z is a constant and which is not equal to 0 in the entire region of the interest.

So, if I look at this function f star of z, we see that it is a very nice function which is an analytic function in this entire region and moreover this f of z has no 0s, right. So, the only way that this function f star of z could have had some mess right or some normal analyticity is if this function f of z took a value 0, but since it does not take a value 0 and since f of z itself is a you know nice analytic function this f star of z is also an analytic function, it is analytic in this entire region of interest.

So, we have managed to show that there is a region in which both f of z and f star of z both the function and its complex conjugate are analytic in this entire region and we have seen from the previous example that the only way that this can happen is if f of z is a constant right that is the result.

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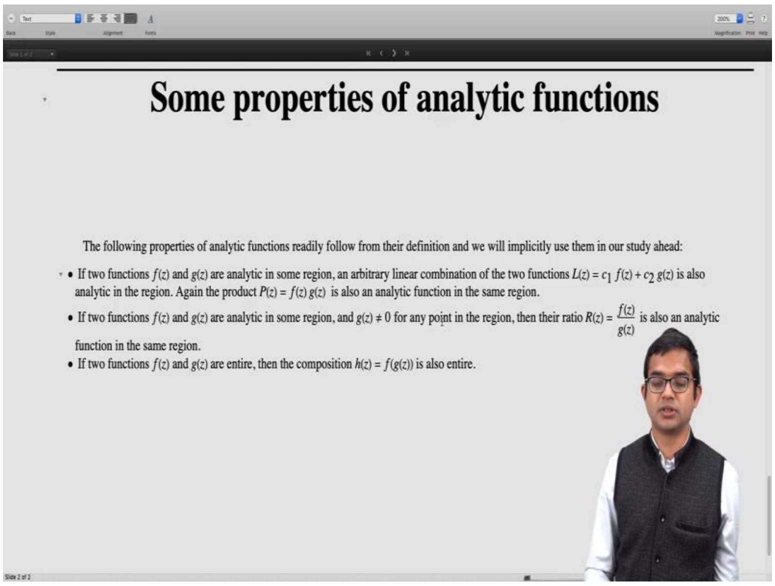

So, let us look at some properties of analytic functions. So, this comes about directly from the definition and we will not try to give a rigorous proof of it or anything like that. We will just state these properties and we will take them for granted and we will use some of these properties implicitly as we go along, right.

So, if there are two functions which are analytic in some region and you take a linear combination of these two functions and form a new function then this new function also is going to be analytic in this entire region. Again, if you take a product of two analytic functions, the product is going to be also an analytic function in the same region of analyticity of each of these functions.

And, if two functions f of z and g of z are analytic in some region you can also take the ratio of these two, right. So, we already use this fact and provided the denominator is not 0 for any point in the region and this ratio is also going to be an analytic function in the same region, right. So, we already sort of used this property when we argued that f star of z is equal to c squared over f of z must be analytic in this entire region since f of z has no 0, right. So, we are formally stating that property here.

And, again if two functions are entire, so they are analytic in the entire complex plane. So, if you take the composition of such functions if you take f of g of z as and define this as h of z. Now, this h of z is also going to be analytic because g of z has a well defined derivative at any point z and g of z is going to take you to some other point in the complex plane and f of that point in the complex plane.

Also, you know the function f has a well defined derivative and in a neighbourhood around that point. So, for sure f of h of z also has a derivative at any other point that we have gone to and also well defined in the neighbourhood. So, indeed the composition is also going to be analytic.

So, in this lecture we have looked at some properties of analytic functions and looked at a few examples where analyticity brings in some nice properties. We will look at more of these properties as we go along in this series. That is all for this lecture.

Thank you.