Mathematical Methods 2
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Complex Variables
Lecture - 10
Cauchy-Riemann conditions in polar coordinates

So, we have seen how the Cauchy-Riemann conditions are crucial, if you are interested in differentiability, so they are necessary conditions; but they are also part of the sufficiency condition. Something more than Cauchy-Riemann conditions are required, but Cauchy-Riemann conditions for sure need to hold for differentiability. So, often it turns out that it's more convenient to work with functions in polar coordinates rather than in Cartesian coordinates.

We are looking at functions of two variables - instead of looking at x, y real part and imaginary part, it's often more convenient to work with r and theta. So, there is a way to formulate Cauchy-Riemann conditions in polar coordinates which is completely equivalent; but it is written in terms of partial derivatives with respect to r and theta and that is what we discuss in this lecture ok.

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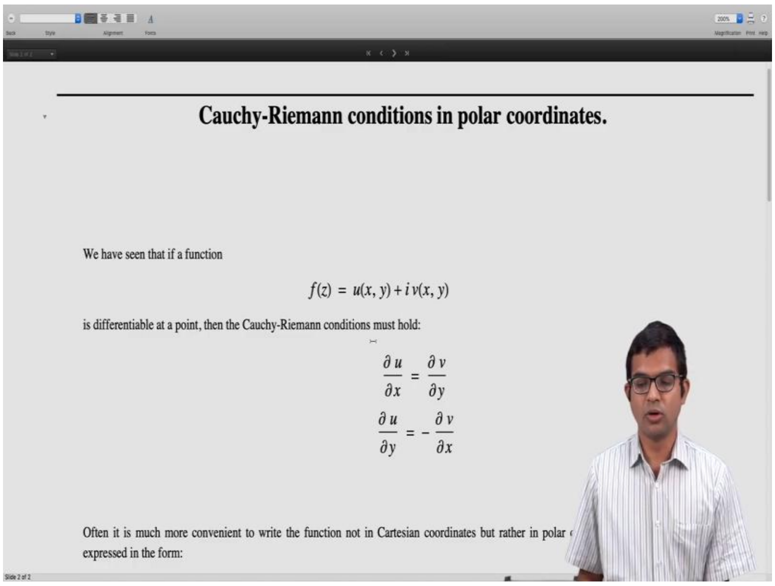

So, the Cauchy-Riemann conditions we all know by now is just dou u by dou x is equal to dou v by dou y and dou u by dou y is equal to minus dou v by dou x, where u and v are the real part and the imaginary part of this function f of z and both u and v are functions of x,y, where x and y are the real part and the imaginary part of the variable z; the complex variable z right. So, if we are interested in recasting this function in terms of polar coordinates, so the function would have a similar form.

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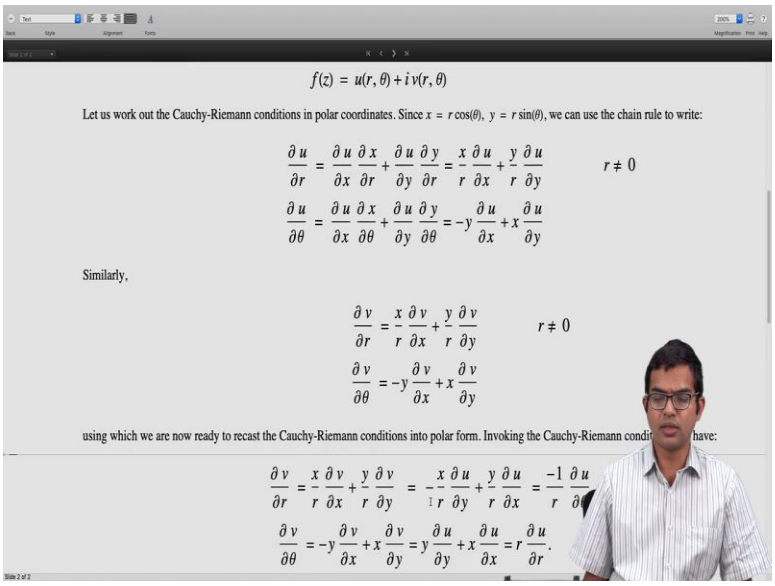

So, now, instead of saying f of z is equal to u of x,y, we could cast it as f of z is equal to u of r, theta plus i times v of r, theta right. So, this function we have written in terms of this real part and an imaginary part. But the real part and imaginary part are taken to be functions of r and theta right, rather than the real part and the imaginary part of the variable right.

So, it's often convenient to use this formulation. So, directly working with Cartesian conditions can become extremely tedious for some complicated functions for which we may want to test Cauchy-Riemann conditions. So, instead, we can directly work with polar coordinates. So, polar coordinates, we work with polar coordinates using these relations x and y. x is given in terms of r and theta as x is equal to r cos theta and y is equal to r sin theta, right.

So, we can use the chain rule to first of all write down dou u by dou r and dou u by dou theta right. So, dou u by dou r is nothing but dou u by dou x and dou x by dou r plus dou u by dou y times dou y by dou r. But dou x by dou r and dou y by dou r, we can directly work out using the relations x equal to r cos theta and y equal to r sin theta. So, dou x by dou r is nothing but cos theta.

But it's convenient to rewrite cos theta in terms of x by r. So, that is what we do here. So, dou x by dou r is cos theta; but cos theta is x by r and likewise, dou y by dou r is sin theta, but sin theta is y by r. So, we write this as x over r dou u by dou x plus y over r dou u by dou x.

Of course, this is valid only if r is not equal to 0 right. So, the origin has to be excluded. So, when you are working with you know polar coordinates and dou u by dou theta is similar except that here you do not have this r not equal to 0 constraint; but overall in any case if you are going to work with polar coordinates in Cauchy-Riemann conditions, we will have to look at points other than the origin.

So, dou u by dou theta is equal to again the same trick, chain rule dou u by dou x times dou x by dou theta plus dou u by dou y times dou y by dou theta and dou dou u by dou x is. So, dou x by dou theta is simply minus r sin theta right. But minus r sin theta is nothing but minus y. So, we write it as minus y times dou u by dou x and then, once again, dou y by dou theta is r cos theta, which is the same as x. So, the second term becomes x times dou u by dou y right.

So, this u is some arbitrary function of r and theta. So, we have managed to show you know these two relations for dou u by dou r and dou u by dou theta. So, even for the function v of r, theta, the same relation or something very similar must hold. So, dou v by dou r, we can immediately write down must be equal x by r times dou v by dou x plus y over r times dou v by dou y and again, r not equal to 0 and dou v by dou theta must be equal to minus y times dou v by dou x plus x times dou v by dou y right.

So, all we have done so far is to simply use the definition of you know what polar coordinates are and we are looking at how partial derivatives you know with respect to the polar coordinates are related to the partial derivatives with respect to Cartesian coordinates. That is all we have done so far. So, now, we will invoke or we will make use of the Cauchy-Riemann conditions right.

So, let us look at how you know the Cauchy-Riemann Riemann conditions change and how they can be written in terms of polar coordinates. So, if we. So, d v by dou v by dou r we have just shown is the same as x by r times dou v by dou x plus y over r dou v by dou y. But from Cauchy-Riemann conditions in Cartesian coordinates, we know that dou v by dou x is the same as minus dou u by dou y.

So, we read this as minus x over r dou u by dou y and then, we also know that dou v by dou y is the same as dou u by dou x. So, this is the other Cauchy-Riemann condition. So, we have plus y over r dou u by dou x. But what is this quantity? So, this whole quantity is something we have already worked out. It's the same and if you look at this equation here, it is the same except that there is an extra minus sign. So, I have a minus and also an r.

So, I have a minus x over r dou u by dou y plus y over r dou u by dou x. So, this can be identified as nothing but minus 1 over r times dou u by dou theta. So, I should look at this and this. I have managed to show that dou v by dou r must be equal to minus 1 over r dou v by dou theta and r not equal to 0.

So, the other condition will come in terms of dou v by dou theta. So, dou v by dou r is one; dou v by dou theta is now we start with this minus y d v by dou v by dou x plus x times dou v by dou y and then, immediately, we invoke Cauchy-Riemann conditions to rewrite dou v by dou x as minus dou dou u by dou y. So, we have a minus sign cancel, then we have a y times dou u by dou y and again, dou v by dou y can be rewritten as dou u by dou x Cauchy-Riemann conditions.

So, we have y times dou u dou y plus x times dou u by dou x; but then, we see that this thing is something we have already seen and we just have to look at this equation and we see that if you multiply this by r, you will basically get this. So, we have r times dou u by dou r right. So, we have managed to find two conditions; one for dou v by dou r and the other for dou v by dou theta expressing them in terms of dou u by r and dou u by dou theta.

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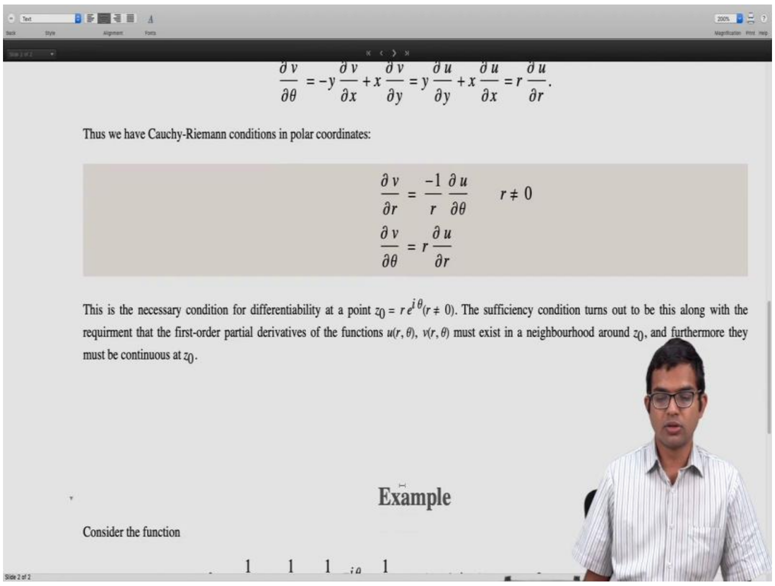

So, these are the Cauchy-Riemann conditions in polar coordinates. So, saying it again, dou v by dou r is equal to minus 1 over r dou u by dou theta and dou v by dou theta is equal to r times dou u by dou r, r not equal to 0 right. So, this is the necessary condition for differentiability at some point as z not equal to r times e to the i theta r not equal to 0. The sufficiency condition again is the Cauchy-Riemann conditions plus you know some nice properties for these functions u of r comma theta and v of r comma theta in the neighborhood.

So, it is not enough, if Cauchy-Riemann conditions hold at that point; but in fact, in a neighborhood around that point, you must have first order partial derivatives and they must also be continuous at that point z0 right. So, if you add these requirements to Cauchy-Riemann conditions, then you also get the sufficiency condition satisfied.

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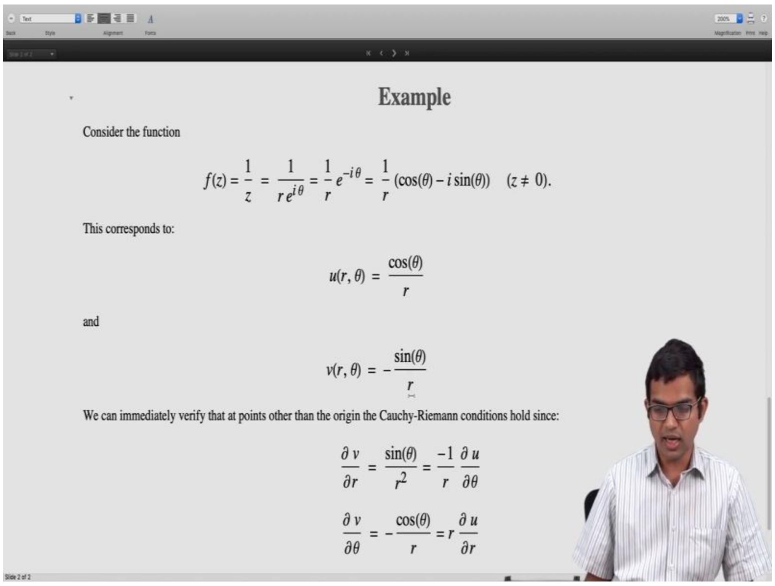

So, let us quickly look at an example. So, if you take some function like f of z, you can make up your own functions involving z and try to work out the derivatives or Cauchy-Riemann conditions in the polar coordinates right. So, 1 over z, suppose we do 1 over z, we get 1 over r times e to the i theta which is the same as 1 over r times e to the minus i theta, which is cos theta by r minus i sin theta by r and z not equal to 0.

So, r not equal to 0. So, u of r comma theta is cos theta by r and v of r comma theta is minus sin theta by r, we can immediately verify the Cauchy-Riemann conditions hold because dou v by dou r is equal to plus sin theta by r square which is a same as dou u by dou theta divided by minus r.

So, dou u by dou theta will reverse minus sin theta by r, then we have to divide by r to get you r square and a minus sign is also there. So, it is exactly the same as this dou v by dou r. Likewise, we can verify that dou v by dou theta which is minus cos theta by r is the same as r times dou u by dou r which will give you minus 1 over r square, then you multiply by r. So, you get back minus cos theta by r.

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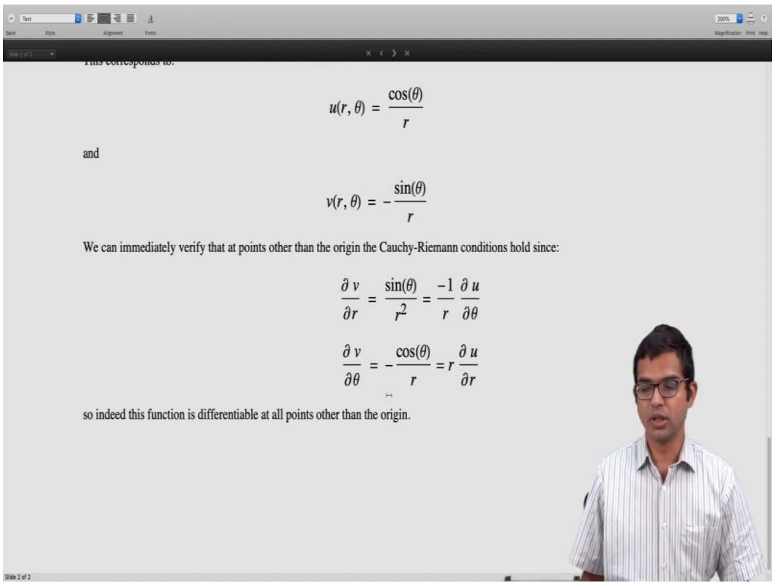

So, indeed, the Cauchy-Riemann conditions hold at the origin and so this function is at all points other than the origin and indeed, this function is differentiable at all points other than the origin right. So, for that you need some more condition; but, so the statement is that the function is differentiable at all other points and therefore, Cauchy-Riemann conditions hold, which we have verified even with polar coordinates.

But you can also verify this directly with Cartesian coordinates if you wish ok. That is all for this lecture.

Thank you.