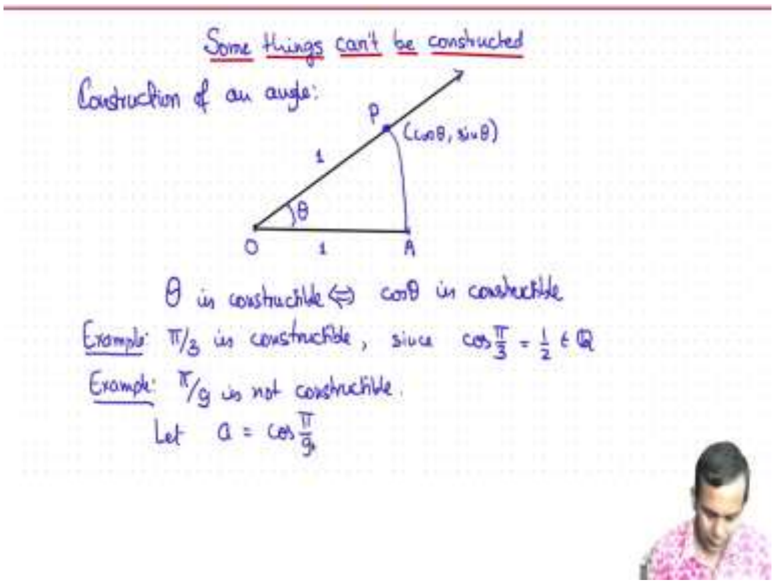
Algebra-II
Professor Amritanshu Prasad
Mathematics
Indian Institute of Sciences
Some things cannot be constructed

(Refer Slide Time: 00:14)



Just as there are some things in life that money cannot buy, there are some numbers in the reals that straightedge and compass cannot construct. We look at some things that cannot be constructed today. We will use the criterion that we discovered last time, that if a real number is constructible, then it generates a field whose degree is a power of 2.

So the most famous example of something that can be constructed is the, we cannot trisect an angle using straightedge and compass. In order to establish this, all we need to do is find one angle which cannot be trisected using straightedge and compass. So in order to do that, let us think about the construction of angles for a moment.

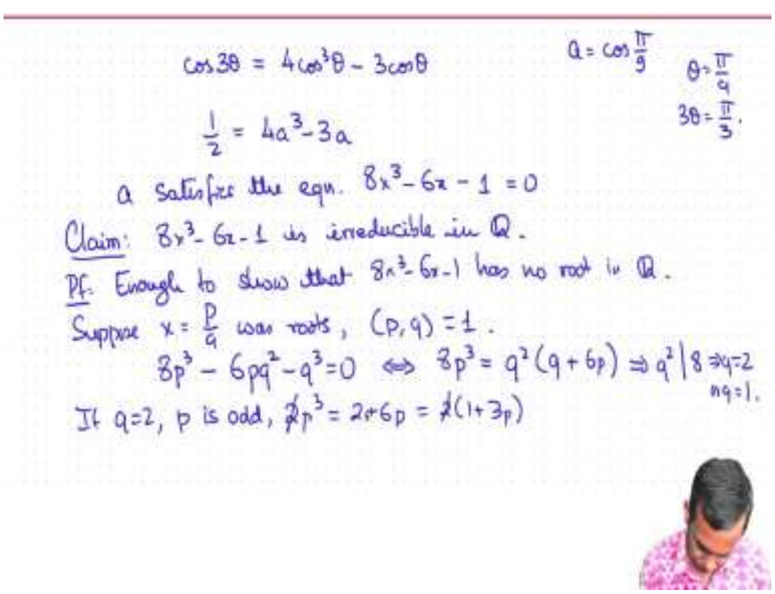
So, construction of an angle, so let us start with our usual point, our usual two points which are given to us namely, O and A. And, let us say we want to construct an angle theta. So we just construct angle theta. Now this distance is unit one. We can also mark it off, along this arc here. So this is also, we let us also take this to be one. And then, this point P here has coordinates, cos theta comma sine theta.

So, if the point P can be constructed then we can also construct the then the, the length, the real number cos theta is constructible. So, P is constructible implies that cos theta is constructible. And, if cos theta is constructible then sine theta square root of one minus cos square theta sin theta in a quadratic extension of the field containing cos theta; so sine theta is also constructible and therefore, since the x and y coordinates of P are constructible, P itself would be constructible. So, this is an effect only if, if construction.

So, what we will say is that an angle theta is constructible, if and only if, cos theta is constructible; because if this angle theta were constructible, we would be able to mark off point at distance one along thusly, and we would be able to construct the point. So, this is a working definition for an angle being constructible, that cosine theta is constructible.

So for example, the angle pi by 3 is constructible. Why? Because, cos pi by 3, pi by 3 is 60 degrees, it is just half which belongs to Q, and every rational number is constructible. But here is the second example; pi by 9 is not constructible. And, let us see why this is not constructible. So, let a be the number cos pi by 9, okay.

(Refer Slide Time: 04:11)



So, we will use the triple angle formula, which says that, cos 3 alpha, three theta is equal to 4 cos cube theta minus 3 cos theta. So now we put, we have a equals cos pi by 9. And so, this is theta. 3 theta is pi by 3. So, what we get is pi by 3, cos pi by 3 which is half, is equal to 4 a cube minus

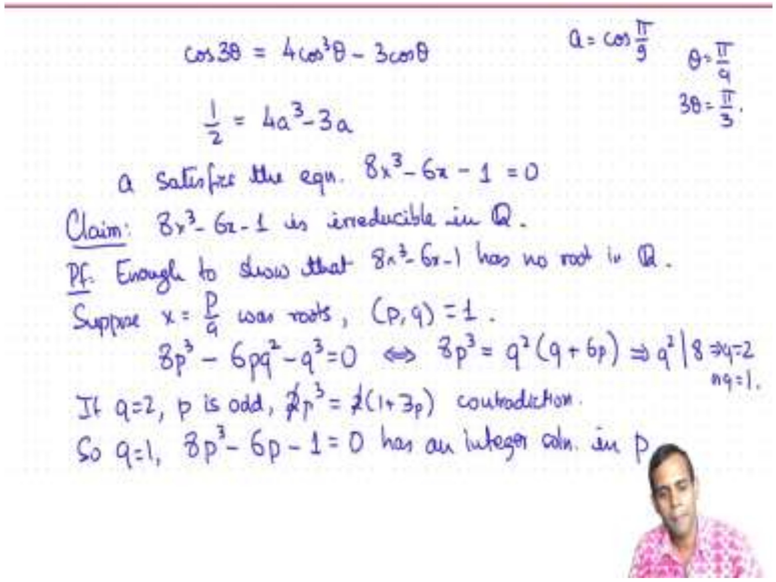
3a. So, what we have is that a satisfies the equation. 8 x cubed minus 6 x minus 1 equals zero. So, it is a root of the polynomial 8 x cube minus 6 x minus 1.

Now, I claim that 8 x cubed minus 6 x minus 1 is irreducible. Now, if it were not irreducible, it would have at least one linear factor because any factorization would either be into a polynomial of degree 2 and polynomial degree 1. Well, there is no other possibility, or 3 three polynomial 1. But it would certainly have a linear factor. So, to prove this, it is enough to show that this has no root. It has no linear factor means that it has no roots in Q.

So, this, when I say reducible I mean irreducible in rationals. So, we need to show that it has no root in Q. So, suppose that we had a root. Suppose we had root, x equals p by q was a root. And, by canceling common factors we can assume that the GCD of p and q is 1. So, the fraction is given in lowest terms. Then what we have is 8 p cubed minus 6 p q squared, just multiplying through by q cubed minus q cubed is equal to zero, which is the same as saying that 8 p cubed is q squared into q plus 6 p.

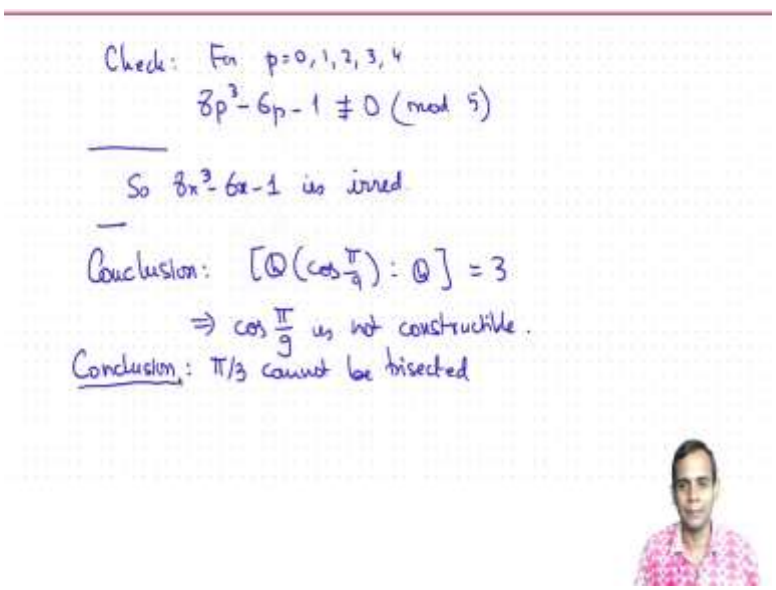
Now, p and q are co-prime. So, this means that the only common factors of q can be with 8. And, so this means that q squared divides 8, which means that q equals 2 or q equals 1, okay. So, if q equals 2, this means that p has to be odd, because p and q have no common factors. And, what we get is 2 p cubed, just you know, is equal to 2 plus 6 p, which is 2 into 1 plus 3 p. So, we cancel out this, we get...

(Refer Slide Time: 07:41)



Maybe we can just move this middle step. So, we get p cubed is 1 plus 3 p. But, p is odd. Then 1 plus 3 p is even but p cubed is odd. So, this is a contradiction. So, it is not possible at q as 2. That means q is one, which means that this 8 p cubed minus 6 p minus 1 equal to zero, has an integer solution in p. But, now you just take residues mod 5.

(Refer Slide Time: 08:38)



Just check for p equals 0, 1, 2, 3, 4. This equation, 8 p cubed minus 6 p minus 1 is not congruent to zero mod 5. If you had an integer solution, it could also be a solution mod 5. But since this equation, you can just do force check that it does not have a solution mod 5. And, since it does not have a solution mod 5, it cannot have an integer solution. So, what we have is that 8 p cubed, 8 x cubed minus 6 x minus 1 is irreducible.

So, conclusion is that the degree of cos pi by 9 will over Q is 3, which implies that cos pi by 9 is not constructible because 3 is not power of 2, right. And, the, this actually implies that not every angle can be trisected because the angle pi by 3 cannot be trisected. So, when we say an angle cannot be trisected it means that you are given the angle and then you have to construct using straightedge and compass one third of the angle.

But, if you are given pi by 3, it is not really that you are given anything because the angle pi by 3 is already constructible. So, no new points are given that you could not have otherwise constructed, right. So, given an angle of pi by 3, it does not really give you anything. And, you cannot construct cos pi by 9, so you can never construct an angle of pi by 9, even if you are given an angle of size pi by 3. So, pi by 3 cannot be trisected. And, so in general, there is no algorithm, using straightedge and compass to trisect, an angle.