Algebra 2
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Krull-Schmidt Theorem

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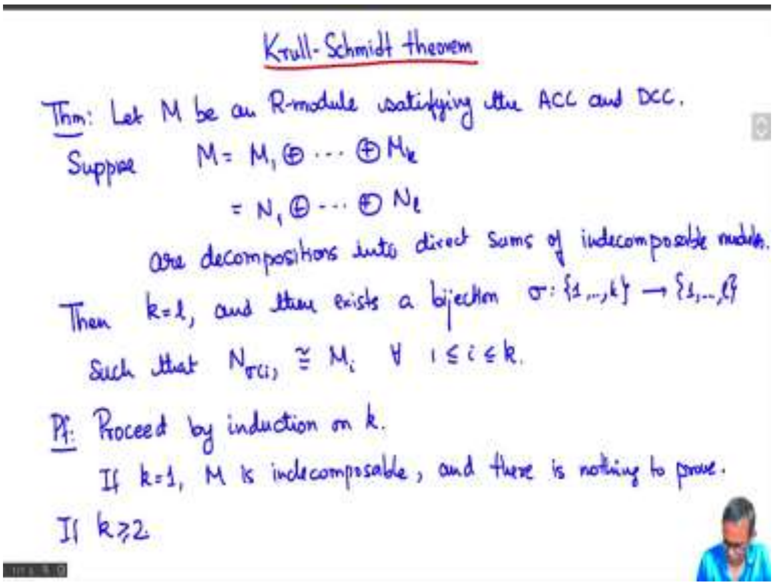

In this lecture, I will prove the Krull-Schmit Theorem. So, the theorem says let M, R can be any ring, let M be an R module satisfying the ACC and the DCC. We know already that such R modules can be written as a direct sum of indecomposable sub modules. And suppose we have two such compositions. Suppose, M equals M1 direct sum Mk and N1 direct sum Nl are decompositions of M into indecomposable sub modules, into direct sums of indecomposable modules.

Then k is equal to l and there exists a bijection, sigma 1 to k to 1 to l such that n sigma i is isomorphic to Mi for all i between 1 and K. In other words, the sum adds in this decomposition are uniquely determined up to the reordering of sum adds. The proof will use the locality of the endomorphism algebra and also use the calculus of projections. So, but, so the proof will proceed by induction on K. If k equals 1, there is nothing to prove. M is in decomposable and so N itself must also have only 1 sum adds. Otherwise, let us assume that k is greater than or equal to 2. Now, we have these decompositions.

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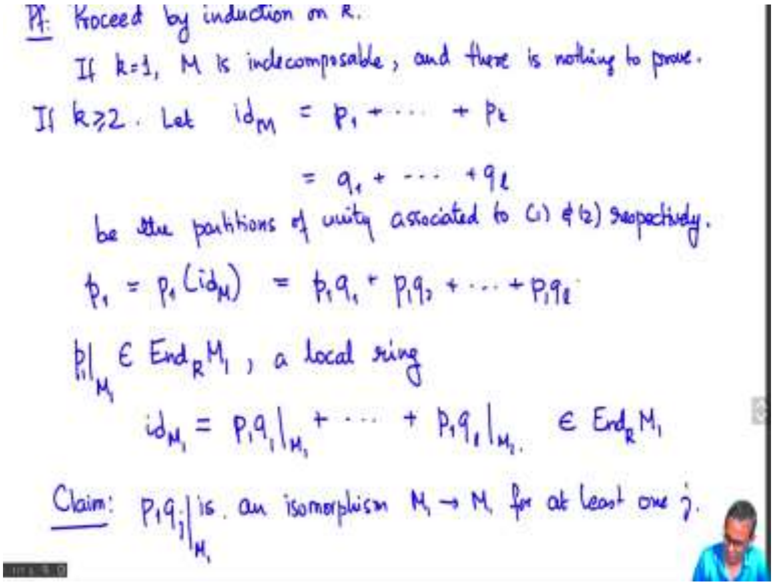

Let identity of M equals p1 plus pk and q1 plus ql be the partitions of unity associated to, the two direct sub decompositions which maybe I can give them names 1 and 2, respectively. So, now let us look at p1. Well p1 is p1 times identity of M, but then identity of them as a decomposition. So, it is q1 plus q2 plus ql. So, p1 q1 plus p1 q2 plus p1 ql.

Now, p1 restricted to M1, p1 anyway its image is contained in M1. So, we can think of people restricted to M1 as an element of end R M1 and by the theorem which I proved in the previous lecture, this is a local ring and what we have is that the identity of M1 equals p1 q1 restricted to M1 plus p1 qn restricted to M1.

Now I claim that p1 qj is an isomorphism or maybe I should say p1 qj restricted to M1 is an isomorphism from M1 to M1 for at least one j. Why is that, well this whole thing is happening inside end R M1 and this end R M1 is a local ring and so, if none of these sum adds where isomorphisms, then by the theorem that we proved in the last lecture, they would all be nilpotent.

And so, but then and there’s sum also would be nilpotent. Because all these nilpotent elements, they form a two sided idea. But then the sum is the identity, which is clearly not nilpotent. So, it is a contradiction. So, p1 qj must be an isomorphism for at least 1 j.

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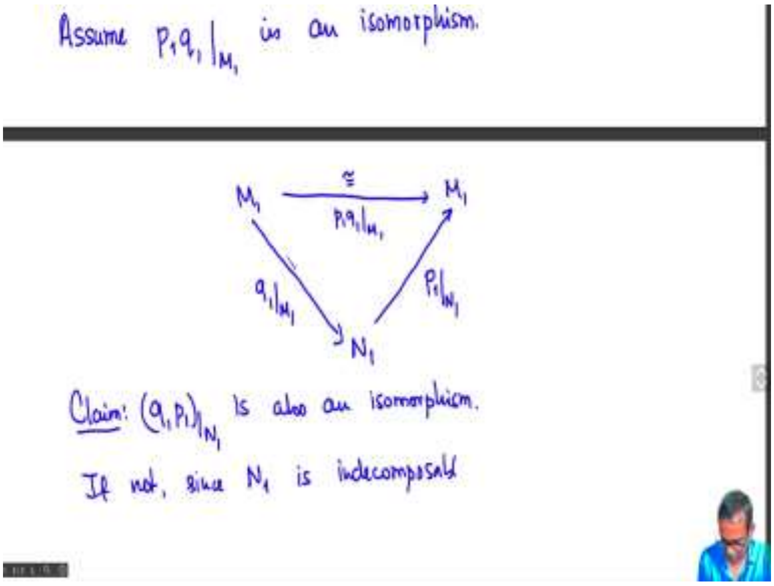

And so let us assume without loss of generality that p1 q1 by renumbering is an isomorphism. So, what we have is we have M1, so I should say p1. So, what we have is we have M1, so I should say p1, so we have M1 and then we have M1 and we have this isomorphism p1 q1 is raise to M1 and we can look at p1. So, q1 goes from, q1 restricted to M1, its goes to N1 because the image of Q1 is contained in N1 and P1 restricted to N1 goes back into M1 and what we are saying as this composition is an isomorphism. Now, I claim that q1 p1 restricted to N1 is also an isomorphism. Why is that?

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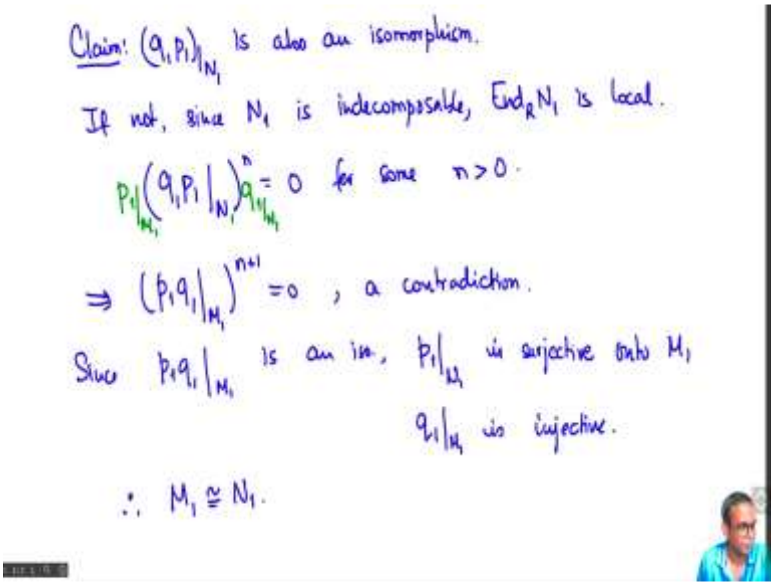

If not since N1 is indecomposable end R N1 is also local and in fact, by the previous the theorem, in the previous lecture what we have is q1 p1 restricted to N1 to the power n is 0 for some N greater than 0. But now what you can do is, you can multiply this on the left by p1 and on the right by q1 and what you will get is p1 q1. So, here you are multiplied by p1 restricted to M1 and here q1 restricted to N1. So, what you will get is p1 q1 restricted to M1 to the power n plus 1 is equal to 0. But that contradicts the fact that p1 q1 is an isomorphism.

So, now since p1 q1 restricted to M1 is an isomorphism we have p1 restricted to M1 is surjective, p1 is surjective on, p1 is surjective on to, p1 restricted to M1 is surjective on to M1 and we also have that q1 restricted to M1 is injective. And therefore, p1 restricted to N1 and q1 restricted to M1 define isomorphisms between M1 and N1. So, M1 is isomorphic to N1.

So, we found from these decompositions, we have been able pick out one sum add from the first decomposition one sum add from the second decomposition that are isomorphic. In fact, we just renumbered the second composition to make that the first sum add. So, what we have is that M1 is isomorphic to N 1. Now, for induction to cook, to kick in it suffices to show that.

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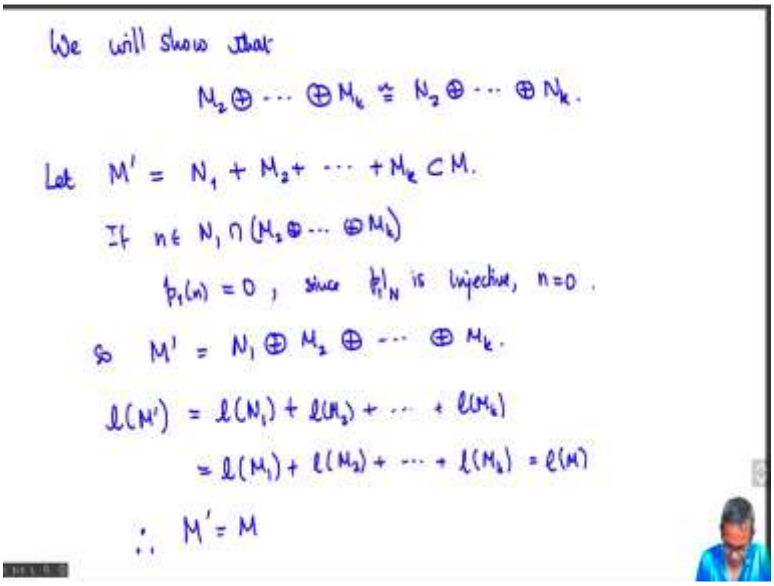

We will show that M2 plus Mk is isomorphic to N2 plus Nk that is not really obvious yet. So, let M prime be just the sum N1 plus M2 plus Mk. What I mean is just, you could just put plus here, but this part of the sum will be direct and this part, this is a sub module of M. Now, if N belongs to N1 intersect M2 plus Mk, then what we have is that p1 n has to be 0 because p1 vanishes on M2 direct sum Mk.

But recall that p1 restricted N is injective, it is an isomorphism on to M1, we get that n is equal to 0. So, this N1 intersect M2 direct sum Nk is 0 and so we have M prime is N1 direct sum M2 direct sum Mk. So, I want to show that M prime is actually equal to M, we can do this by comparing lengths. Length of M prime is length of N1 plus length of M2 plus length of Mk.

If you have a Jordan-Holder series for each sum add you can combine them to get a Jordan-Holder series for the direct sum and this is equal to well N1 is isomorphic to M1. So, the length of N1 is equal to the length of M1 plus length of M2 plus length of Mk that is equal to the length of M. So, since M prime is a sub module of M with the same length as M prime, M prime must be equal to.

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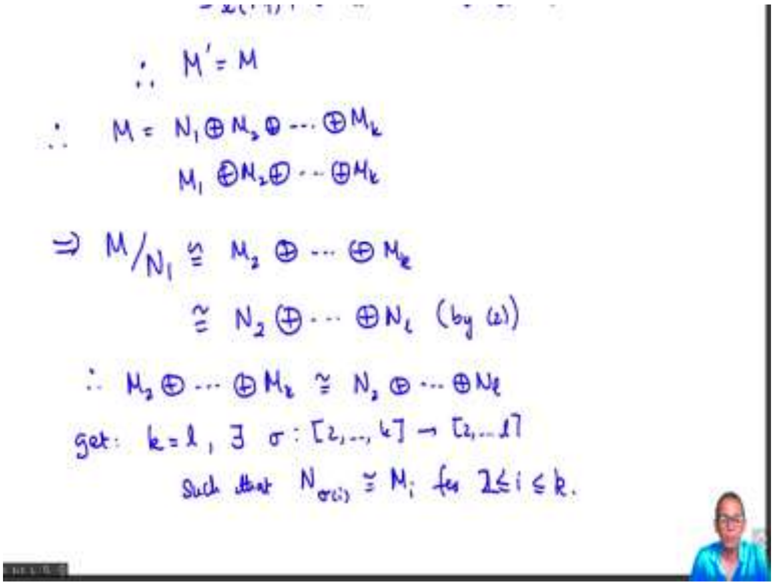

So, what we have is, m is N1 direct sum M2 direct sum Mk and M is also M1 direct sum M2 direct sum Mk. So, it follows that M mod N1 is isomorphic to M2 direct sum Mk, but also by the decomposition 2 that we had up here, it follows that M mod N1 is isomorphic to N2 direct sum Nl.

By the decomposition 2 and therefore, M2 direct some Nk is isomorphic to N2 direct sum Nl, so the induction hypothesis on k kicks in and, so what we get is k minus 1 is equal to l minus 1 which is the same as saying that k equals l and there exists a bijection from 2 to k 2 to l, such that N sigma i is isomorphic to Mi, for each i, for 2 less than or equal to i, less than or equal to k and putting the act together with the fact that M1 is isomorphic to N1 proves the theorem for Krull-Schmit for M.