Algebra- II
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Lecture 80
Direct Sum Decompositions

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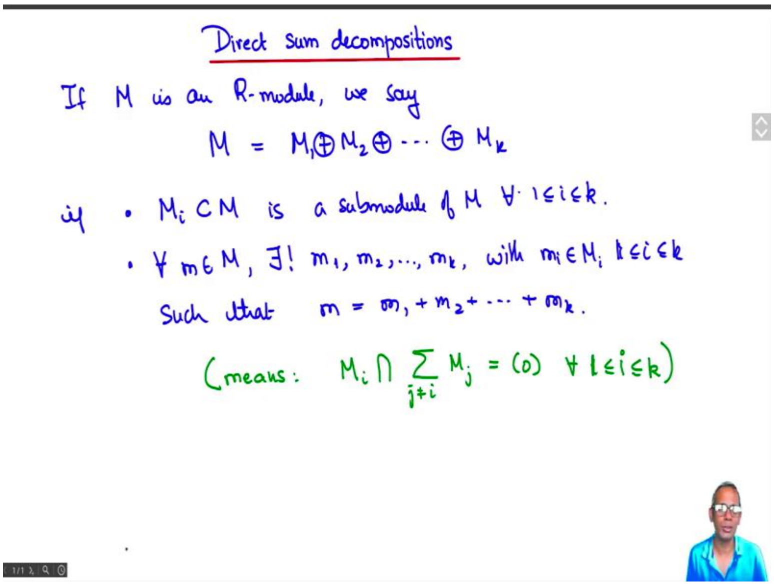

In this lecture, we will look at direct sum decompositions in greater than generality, than we did in the previous lecture. So, now, suppose, M is in R module, we say that M is the direct sum M1, M2 Mk. If firstly Mi is a sub-module of M for every i, let us say every i between 1 and k. And secondly for every element m of M, there exists unique elements, m1, m2, mk with mi belonging to mi or i between 1 and k such that m is equal to m1 plus m2 plus mk.

So, this is similar to the decomposition into two parts, but then we also had this condition that m1 intersect m2 is trivial being equivalent to the uniqueness. And in this case what this means is that, if you take Mi and intersect it with the sum of all the other modules, except i then this will be 0 for every i in between 1 and k.

So, that is the analog of the intersection 0 condition. But in any case, what we are saying is that M has a unique decomposition. So, you should try as an exercise to prove that the uniqueness of this decomposition is equivalent to this. And the proof is quite similar to the proof in the case where we just had a decomposition into two sub modules.

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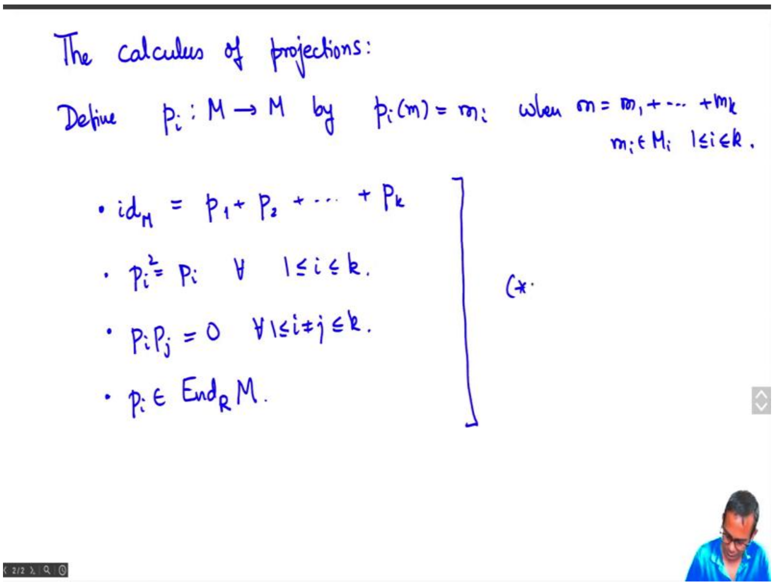

And now we can do the calculus of projections in greater generality. So, again, you define pi from M to M by pi of m is equal to mi when m has decomposition m1 plus mk with mi belongs to Mi for 1 less than or equal to i less than k. And of course this is well defined because the mis are uniquely determined by m. And once again, we have the following conditions, the identity element of m is p1 plus p2 plus pk we have pi square equals Pi for every i and then we have pi pj equals 0 for all i not equal to j between 1 and k.

And also we have that Pi belongs to the R R module homomorphisms from M to M which is a ring which we are denoting by EndRM. So, all these things appear are happening inside the ring EndRM. It turns out that this calculus of projection is actually equivalent to a direct sum decomposition. So, conversely let us call these conditions collectively star.

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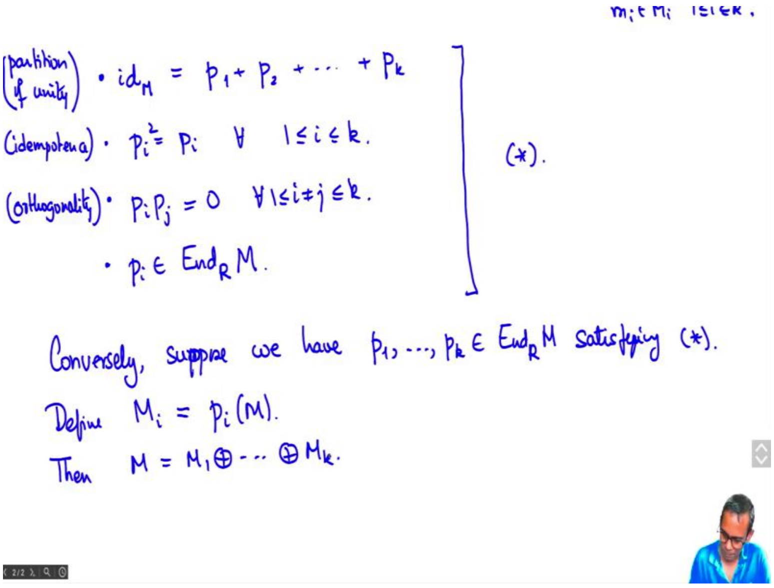

Now, conversely suppose we have elements p1, p2, pk belonging to EndRM satisfying star. So, I just have a module M and I give you elements p1, p2, pk which add up to the identity each Pi is its own square. Which in other words means that pi is an idempotent. pi, pj is 0 for i not equal to j. So, this is called the orthogonality condition.

So, this is to orthogonality, this is the idempotence condition and this condition is sometimes called partition of unity. Suppose we have p1, p2, pk satisfying these three conditions, then define Mi to be the image of pi then we have M is M1 plus Mk. And this again is not very difficult to prove, the fact that the Mi is actually a sub-module follows from the fact that pi is an R module homomorphism.

So, that is not very difficult. And then the fact that each M can be written uniquely as a sum of m1 plus m2 will follow from the two, from the other axioms, the other three axioms, that is partition of unity, idempotence and orthogonality. It is a very nice exercise which I strongly recommend that you all try out.

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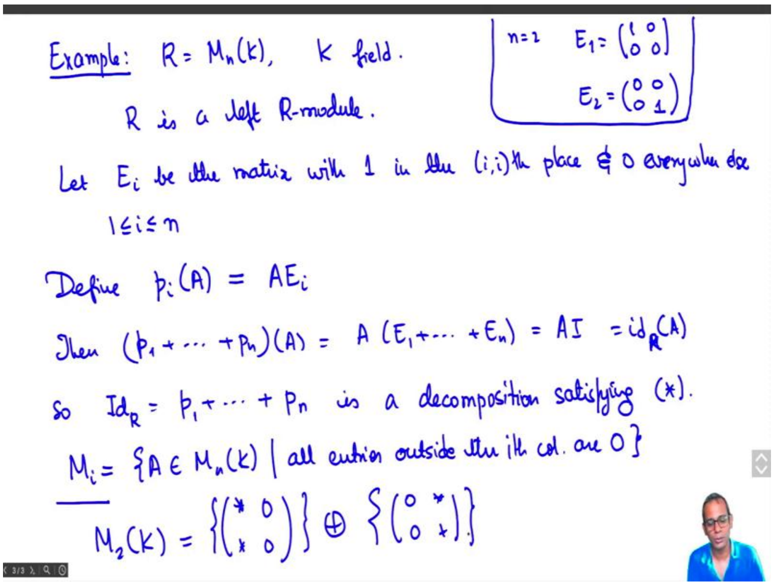

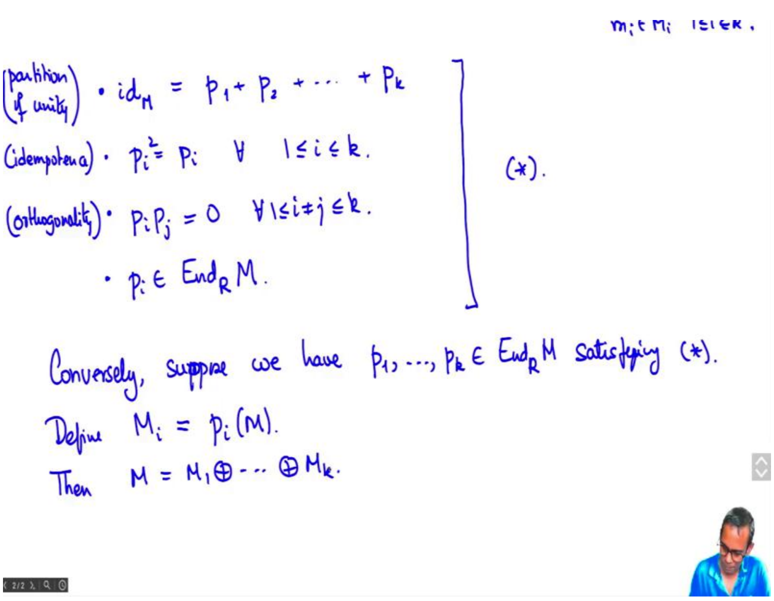

So, let us look at an example of this partition of unity. Let us, take R to be Mn K and where K is a field. And R is a left R module. And let Ei be the matrix with 1 in the i, i th place and 0 everywhere else. So, for example, if n equals 2, I will have E1 is the matrix 1, 0, 0, 0; E2 is the matrix 0, 0, 0, 1. In general, you will have these form, 1 less than or equal to i less than or equal to n.

Then what we have is, the identity homomorphism of M. So, what we will do is we will define a pi of A. So, pi is from Mn K to Mn K it is supposed to be an R module homomorphism. So, let us just define this to be A Ei, then p1 plus pn of A is A is A E1 plus En which is A times identity because E1, E2, En add up to the identity matrix which is the identity of Mn K, identity of R as a left R module also it is easy to see that each pi is in R module homomorphism and so these pis form a partition of unity and they are idempotence and they are orthogonal.

This is identity R of A. So, it satisfies all those conditions on the previous page which I collectively called star. And what is Mi in this case? So, it turns out that Mi is all matrices with non-zero entries only in the ith column or all entries outside the ith column are 0. And indeed, you can see that R will be a direct sum of these sub modules. So, with example, in the two by two case, we have M2 K is the direct sum of matrices with nonzero entries in the first column and matrices with nonzero entries in the second column and these are actually left R sub modules M2 K.