Algebra- II
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Lecture 79
Indecomposable Module

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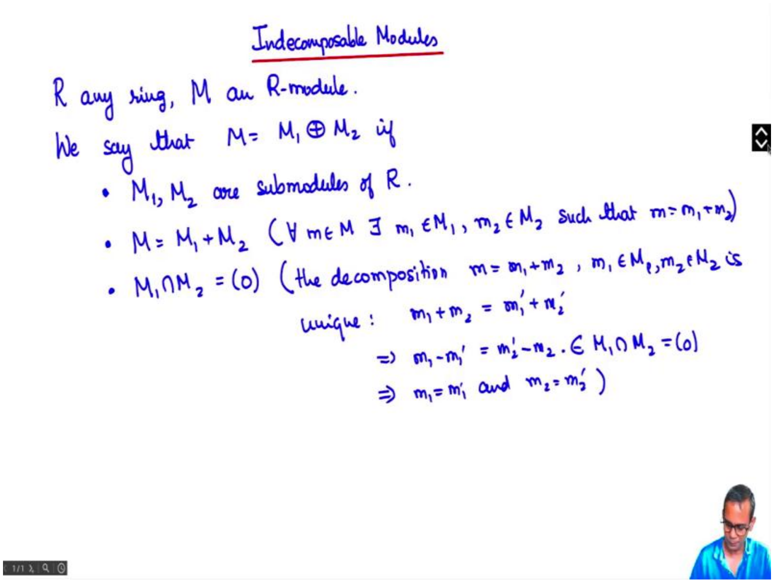

When studied the Jordan-Holder decomposition, we saw how modules that satisfy the ascending chain condition and the descending chain condition are built up in a certain way of modules of a specific kind called simple modules. These are modules which do not admit non-trivial proper sub-modules.

There is another way of breaking up modules into building blocks. And that is into indecomposable modules. So, in this lecture, I am going to introduce you to you the notion of an indecomposable module. So, let R be any ring, M an R module. We say that M is equal to M1 direct sum M2, if the following conditions are satisfied. Firstly, M1, M2 are sub modules of R.

In other words, they are abelian subgroups of the additive group of M and they are closed under the operations from R. When I say R module here, I am talking about a left R module, then the second condition is that M is equal to M1 plus M2 has an abelian group. So, what this means is that, for every m in M there exist m1 in M1, m2 in M2 such that m is equal to m1 plus m2 every element of m can be written as a sum of an element of M1 and an element of M2.

And the third condition is that M1 intersect M2 is a trivial module. And this is equivalent to saying that the decomposition m equals m1 plus m2 is unique. Let me, explain why since it is quite simple. Suppose, we have m1 plus m2 is equal to m1 prime plus m2 prime then this implies that m1 minus m1 prime is equal to m2 prime minus m2.

Now, this is an m1 because M1 prime are in m1, this is an M2 because m2 and M2 prime are in M2. So, these both these things belong to M1 intersect M2, but that consists of only the element of 0, which means that m1 is equal to m1 prime and m2 is equal to m2 prime. So, this third condition means, that the decomposition of M as a sum of elements from M1 and M2 is unique.

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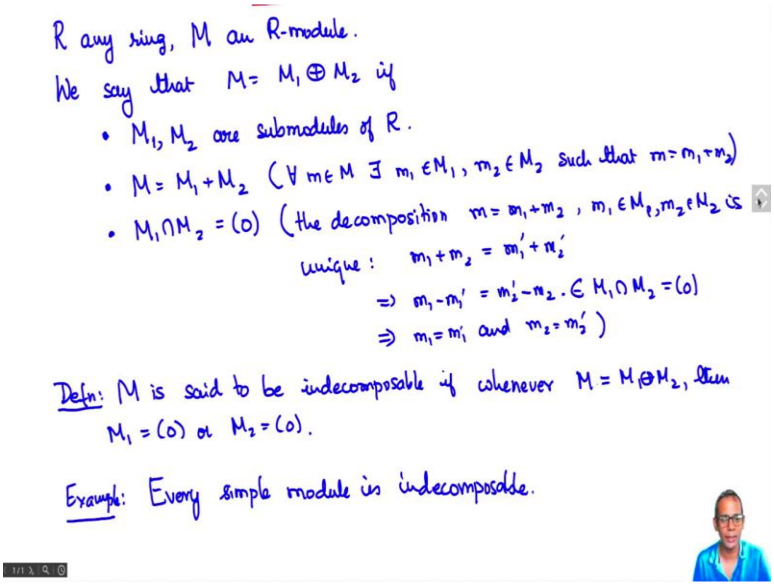

And now, we can define an indecomposable module. M is said to be indecomposable, if, whenever, M is written as M1 direct sum M2, then either M1 is 0 or M2 is 0. Every sub module can be written M can be written as 0 plus M or M plus 0. But an indecomposable module is one that cannot be broken up into two non-trivial summands.

For example, every simple module is indecomposable. Why is that? Well, if we write M as M1 direct sum M2, where M1 and M2 are both non trivial, then both M1 and M2 are non-trivial proper sub modules of M. And so M cannot be simple if it is not indecomposable. Now, there is a nice way of thinking about direct sum decompositions, which will be very useful in the next few lectures.

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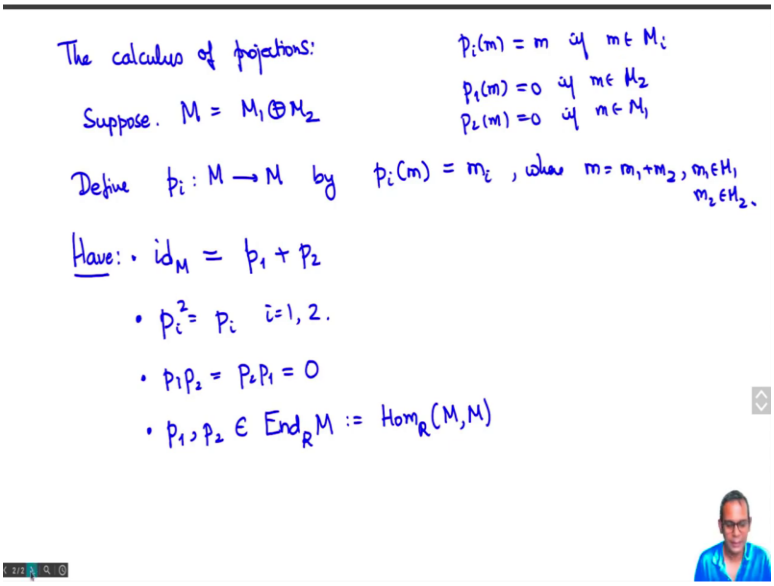

So, let me introduce that and this is sort of the beginnings of the calculus of projections. So, now suppose, so maybe I will just call this the calculus of projections. So, suppose we have M equals M1 plus M2, then what you do is define Pi from M to M as follows. So, if you have an element m, then you can write it as m1 plus m2 in a unique way, with m1 in M1 m2 in M2. And so you just take P1 of m to be m1 and P2 of m to be m2, then what we have are the following properties of these things.

So, you have the identity map of M, it goes from M to M, this is equal to P1 plus P2, that is easy to see, because m is m1 plus m2. So, P1 m plus P2 m is actually equal to m which is the identity map applied to M. The second property is that Pi square is equal to Pi for i equals 1, 2. Each of these Pis is its own square, because once you apply P1 of m you already get m1 and then when you apply P1 of m to an element of m1 it will be written as m1 plus 0 and so, it is itself.

So, the point is that Pi of m is equal to m if m is already in Mi and because of that Pi squared is Pi and the third property is that P1, P2 is equal to P2, P1 is equal to 0. And this is because, P1 of m is equal to 0 if m belongs to M2 and P2 of m is equal to 0 if m belongs to M1. So, both these are easy to check from the definition. And now, if you take P2, you take any element M and then apply P2 to it, you end up in M2 and then when you apply P1 you will get 0.

And most importantly, P1 and P2 are R module endomorphism. What is the meaning of EndRM? This is defined to be the set of all R module homomorphisms from M to M, this

EndRM itself forms a ring addition is point wise addition and multiplication in the string is composition of homomorphisms.

If you have two homomorphisms from M to M, you can compose them with each other in any order that you like. So, in some sense, all these things, which we have written are happening inside this ring EndRM. Now the fact that P1 and P2 are EndRM a rest on the condition that m1 and m2 are sub modules of M. I leave it as an exercise to you to check that because m1 and m2 are sub modules of M P1 and P2 are R module homomorphisms. As an example, as an application of this calculus of projections. Let me, show you an a module that is not simple but is indecomposable

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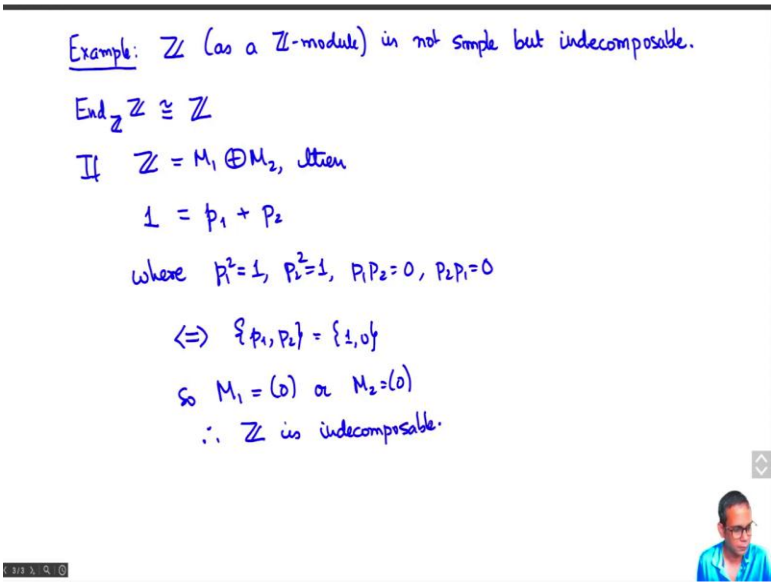

Z as a Z module is not simple. We know that, there are lots of sub modules of Z. For example, multiples of any integer N form a sub module of Z, but it is indecomposable. And one way to see this is to use the calculus of projections. So, firstly, let us compute EndZZ. Well, it turns out that if you have Z module homomorphism from Z to Z it is completely determined by the image of the unit of Z namely the integer 1. If you know where 1 goes, then you know where everyone else goes. And so this, but 1 can go to any integer and so and so EndZZ is isomorphic to Z.

And further, you can check that the operation of composition corresponds to multiplication of integers. And the operation of point wise addition, well, that is more obvious, corresponds to addition of integers. So, as the ring EndZZ is isomorphic to Z. So, if we had, if Z is M1 plus M2, then what we would have is 1 in Z, that is the identity map from Z to Z would be equal to P1 plus P2.

Where P1 square is 1, P2 squared is 1, P1 P2 is 0, P2 P1 is 0, of course this follows because Z here is a commutative ring. But these equations only mean that one of the P1 is 1 and the other is 0. So, the set P1, P2 is equal to the set 1, 0. Which means that either M1 is 0 or M2 is 0. Which is the definition of indecomposable.

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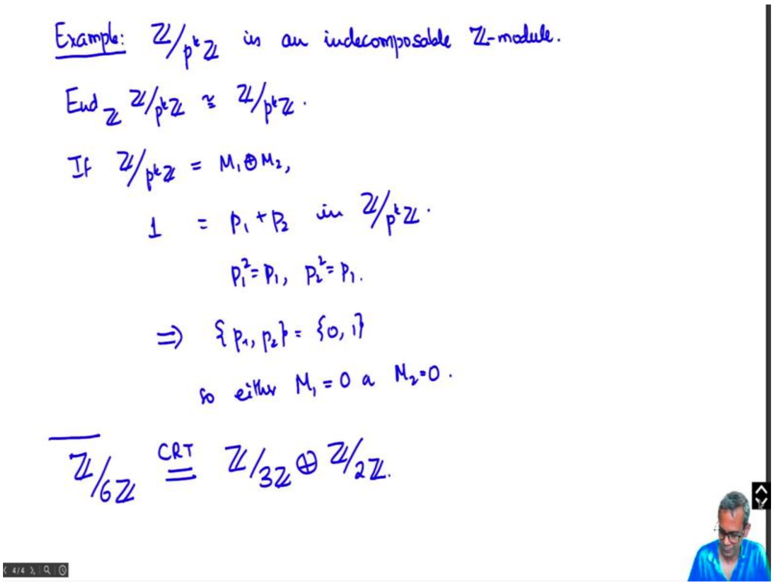

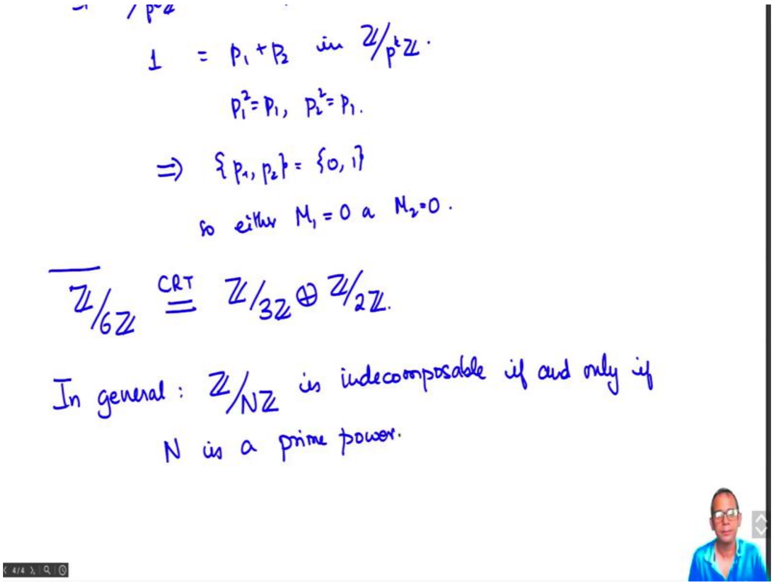

Let us, look at another example. Z mod P to the k Z is an indecomposable Z module. So, again, you show just as before, that the endomorphism ring of Z mod P to the k Z is isomorphic as a ring to Z mod P to the kZ. And so if Z mod P to the kZ were had a decomposition M1 plus M2 and we call P1 and P2 the corresponding projection operators, then we would have 1 equals P1 plus P2 in Z mod P to kZ. Where P1 square is equal to P1 and P2 square is equal to P1. Again, you can check that the only elements in Z mod P2 the kZ that are equal to their squares are 1 and 0. Just as before, this needs a little check.

And so either M1 equals 0 or M2 equal 0. And so Z mod P to the kZ is indecomposable. However, if you take for example, Z mod 6Z, then this by the Chinese remainder theorem is isomorphic to Z mod 3Z direct sum Z mod 2Z as a Z module, and therefore it is not indecomposable. In fact, in general, Z mod NZ is indecomposable if and only if N is a prime power.