Algebra – II
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The Jordan-Holder Theorem for Groups

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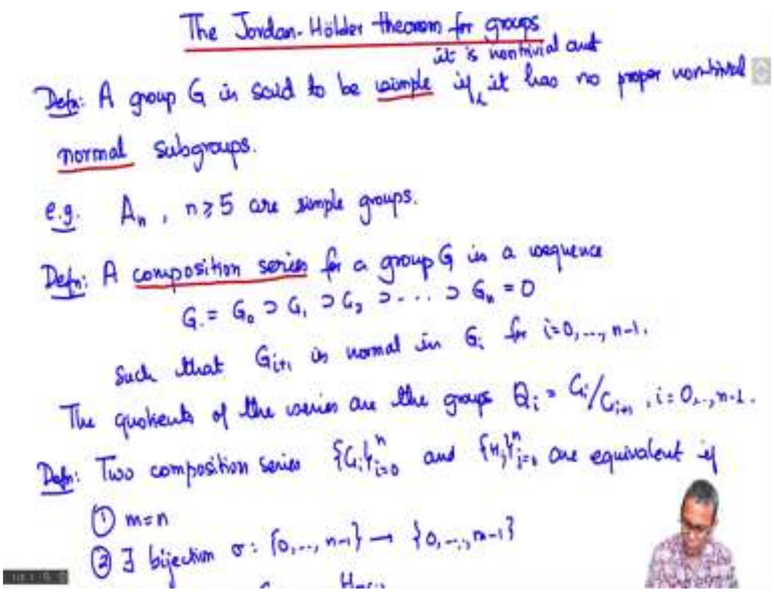

The Jordan-Holder theorem that we have been studying in the last few lectures is what is known as the Jordan-Holder theorem for modules. There is a very similar theorem, which also holds for groups. So, I am going to take a few minutes to explain this variant of the Jordan-Holder theorem. The Jordan-Holder theorem for groups are most of the ideas in the proof are very similar and so, I will not go over the proofs again.

However, working with groups creates a few difficulties, which need to be addressed, I will just tell you what they are. So, firstly, let us come to the definition of a simple group. So, we said that a module is simple if it does not admit any non-trivial proper submodules for a group, we need to be a little more careful. The main point being that, if you have a group and a subgroup, you cannot always define a quotient.

In order to define the quotient of a group by a subgroup, you need the subgroup to be normal. That is when multiplication on the group descends to a well-defined multiplication on the quotients. So, a group is said to be simple, if it has no proper non-trivial, non-trivial means, the trivial group is just the group with only the identity element, no proper non-trivial and the most important word here is normal subgroups.

So, the most common example of simple groups or the alternating groups An for n greater than or equal to 5. And these you can see do have subgroups, any element of such a group will generate a cyclic subgroup, but there are no normal subgroups in these groups. And so, if a group is simple, then it does not have any non-trivial proper normal subgroup, group that is not simple, you can find a non-trivial proper normal subgroup and then you can look at that normal subgroup and maybe you can try to look for each normal subgroup or you can look for normal subgroups in between the group and the normal subgroup.

But in this way, you can construct the analogues of what we had called composition series, this is also called composition series for groups. So, definition again, we need to make a slight change. So, a competition series for a group G is a sequence of subgroups which I will call G equal to G0 containing G1 containing G2 pulling up the Gn, which we will assume to be 0.

Such that and now we need to add this normality condition Gi plus 1 is normal in Gi for each i going from 0 to n minus 1. Now, because of this normality condition, we can define the quotients. So, the quotients of the series are the groups Qi equal to Gi modulo Gi plus 1 for i goes from 0 to n minus 1. Again, this makes sense because Gi plus 1 is normal inside Gi. And now, we can also define the equivalence of composition series.

So, we will say that, two composition series Gi, i equals 0 to n and Hj, j equals 0 to n are equivalent and this definition is identical to the definition in the case of modules, if firstly m is equal to n.

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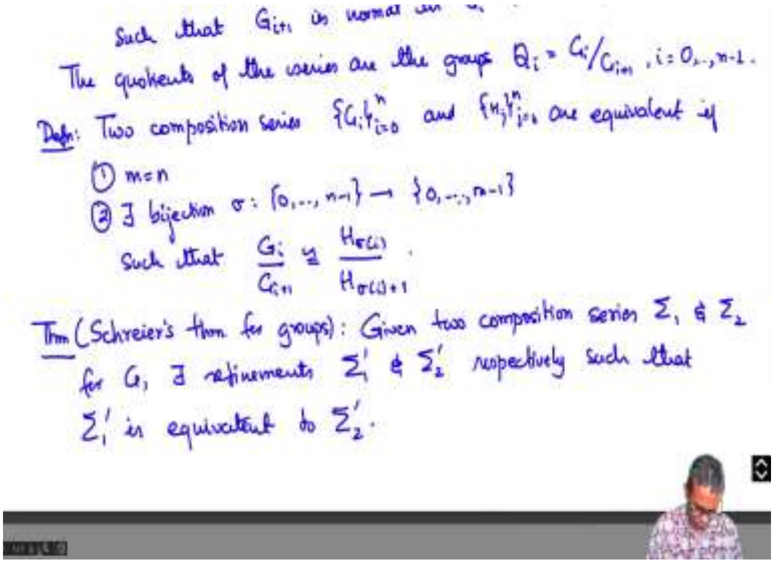

And secondly, there exists a bijection sigma from 0 to n minus 1 on to 0 to m minus 1. Such that Gi mod Gi plus 1 is isomorphic to the group H sigma i mod H sigma i plus 1. And with this, with these slight modifications in place, a Schreier's theorem still holds.

So, we have Schreier's theorem for groups, and it says that given to composition series, sigma 1 and sigma 2 for the same group G, there exists refinements sigma 1 prime and sigma 2 prime of sigma 1 and sigma 2 respectively. So, maybe I will just say respectively, such that sigma 1 prime is equivalent to sigma 2 prime. The proof of Schreier's theorem is very similar to the proof of Schreier's theorem for modules, we just need to be careful to make sure that whenever we take quotients, the subgroups involved are normal.

So, you need to make that check at each stage. But it is not difficult and I am sure if you were to, and I would strongly advise you to try to write down a proof of Schreier's theorem for groups by yourself. So, the argument is exactly the same, you define sigma 1 using sigma 2 and you define sigma 2 using sigma 1. And then we can talk about the Jordan-Holder theorem for groups.

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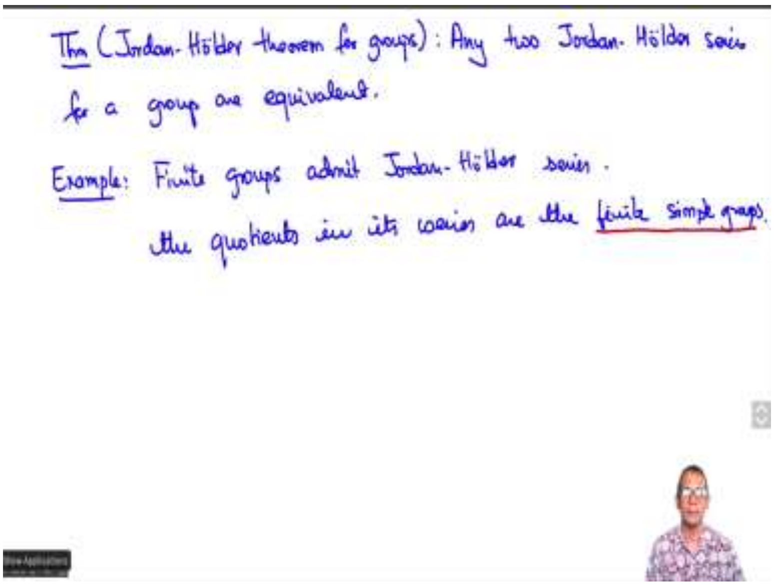

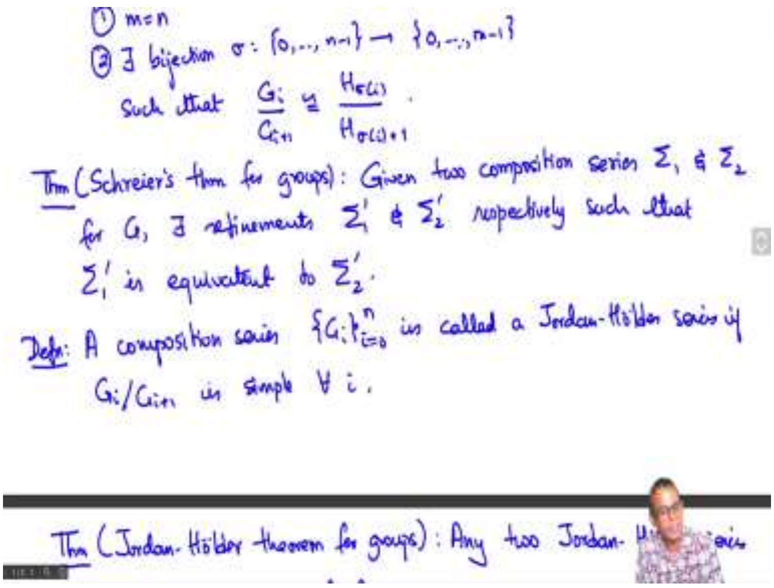

So, this says that, any two Jordan-Holder series for a group are equivalent. This could have been stated for modules also in this way, we are not concerned with whether there exists a Jordan-Holder series or not, we are saying if there are two Jordan-Holder series, which means that they exist and they are equivalent. And this is direct consequence of a Schreier's theorem.

But hang on, I did not explain to you what the Jordan-Holder series is, but I am sure most of you can easily guess. So, let us just define the composition series Gi, i goes from 0 to n is called Jordan-Holder series. If it does not admit a strict refinement, or but maybe I just stated another way if Gi mod Gi plus 1 is simple for every i and this implies that the Gi's are distinct. So, one more thing maybe I should say is that a group is simple if it is non-trivial. And if it is non-trivial and it has no proper non-trivial normal subgroups.

So, we do not consider the trivial group to be simple. So, it is called Jordan-Holder (())(10:22), this means that the Gi's are distinct and that you cannot further refine this composition series by inserting new terms in between. And so, just like before this, these are composition series, which do not admit strict refinements. And so, now, the Jordan-Holder theorem says that any two Jordan-Holder series for a group are equivalent.

And the proof is exactly the same as for modules. And so, just philosophically, what does this say? So, in some sense, it says that. So, let us just talk about finite groups. Because the finiteness, finite groups always admit, Jordan-Holder series. But there are infinite simple groups and there are infinite groups, which admit Jordan-Holder series as well. And so, every finite group has a Jordan-Holder series, where the quotients are finite simple groups.

And this is a very important class of groups because of this theorem, it says that finite simple groups are the building blocks of all finite groups. However, the story does not end there, we do today have a classification of all finite simple groups. However, we are far from having a classification, classification of all finite groups, because even if we know the finite simple groups in the Jordan-Holder series, we do not understand all the different ways of putting them together to construct groups up to isomorphism.

So, it breaks down the problem of classification of finite groups up to isomorphism into two parts, one is the classification of finite simple groups acquired some morphism, which although a major mathematical endeavour, spanning several decades and many mathematicians is has been solved. And the other part is how do you put together finite simple groups are given us a sequence of finite simple groups?

How do you construct all the isomorphism classes of finite groups, which have that sequence as its or that set maybe as a set of quotients for Jordan-Holder series So, we do not know how to answer that second question. But nevertheless, the Jordan-Holder theorem for finite groups is a major theorem in group theory.