Algebra – II
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Examples related to the Jordan-Holder Theorem

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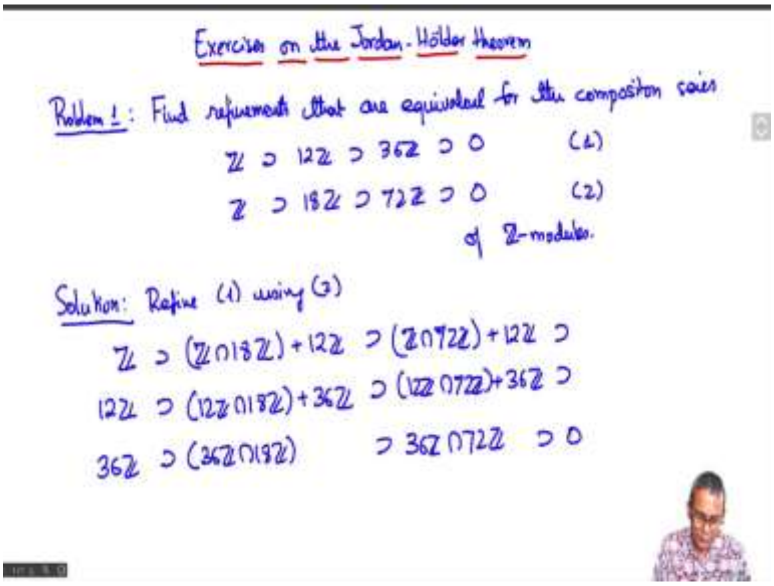

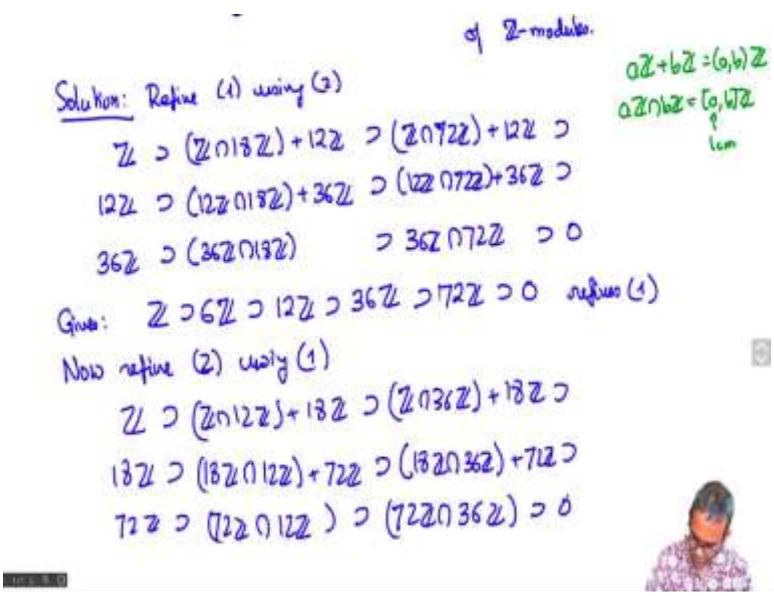

Let us solve some exercises related to the Jordan-Holder theorem. So, here is problem 1. So, I will give you two composition series for the Z-module. And you have to find refinements that are equivalent for the composition series. The first one is Z contains 12Z contains 36Z contains 0, and the second one, so, we call this number 1 and number 2 is Z contains 18Z, which contains 72Z, which contains 0 this is number 2, and these composition series of Z-module. So, you should pause the video and try to solve this yourself.

We will be using the argument that we used in the proof of Schreier is theorem. So, let me solve this for you. So, the idea is to refine each of these series using the other one. So, firstly, let us refine 1 using 2. So, refining 1 using 2 means that, first we will take this part between Z and Z mod 12Z, and we will find the terms in between this corresponding to this filtration. So, what we do is first we take Z and then we take Z intersect 18Z plus 12Z. So, this is something in between Z and Z mod 12Z.

And the next thing in between Z and Z mod 12Z is Z intersect 72Z plus 12Z, once again. And then the last thing is Z intersect 18Z plus 0 which is just Z intersect 18Z. So, I will write that on the next, Z intersect 0 plus 12Z, which is just 12Z. So, I will write that on the next slide. And now we go down and find terms between 12Z and 36Z again using Z 18 and 72Z.

So, what I have to do is now I will take 12Z intersect 18Z plus 36Z, and then I take 12Z intersect 72Z plus 36Z, and then I will get here 36Z. And now 36Z intersect 18Z plus 0, containing 36Z intersect 72Z and that finally contains 0. So, now let us write out the actual terms of the series. We will use two facts, one is that if you have a Z plus b Z, then this is the GCD of a b type Z and a Z intersect b Z then this is the LCM of a b times Z.

So, here, this denotes the LCM. So, using those, we can simplify these terms here. And so, what we get? So, first time is of course Z, then this is Z intersect 18Z, so we have to take the, that is just 18Z, 18Z plus 12Z. So, that is the GCD of 12 and 18, which is 6Z. And now, let us do it here. So, this Z in the sec 72Z is just 72Z. And now we have to take 72Z plus 12Z. So, that is the GCD of 12 and 72, which is 12. So, this is 12Z. And of course, that is still 12Z there.

And so, the next step 12Z intersect 18Z. So, that is the LCM of 12 and 18, which is I believe 36, so 36Z plus 36Z. So, that is just 36Z because the GCD of 36 and 36 is still 36. And then, we have 12Z intersect 72Z, well, that is 72Z, but 72Z plus 36Z is 36. So, we still have 36Z, I will not write it again. And so, again, 36Z over here. Then here we have 36Z intersect 18Z, which is again, 36Z. And then finally, 36Z intersect 72Z, which is 72Z.

So, I am not written the repeated terms, because they will not clearly be needed to write down the common refinement. So, this is the first, this is a refinement of 1. And now, let us refine 2 using 1. So, let us start again with Z. Then we have, we are going to take plus 18Z now. So, we will take Z intersect 12Z plus 18Z, then this contains Z intersect 36 plus 18Z.

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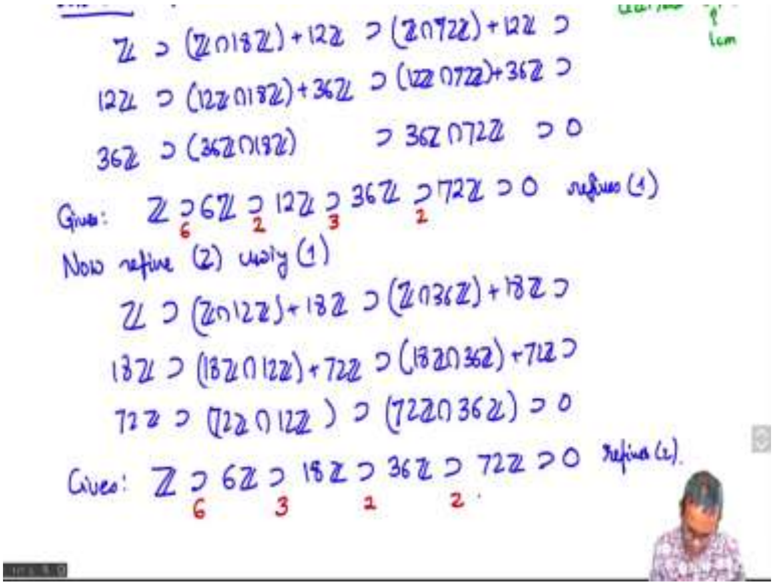

And now this contains 18Z itself, which in term contains 18Z intersect 12Z plus and the next term which is 72Z, and then that contains 18Z intersect 36Z last 72Z. And then finally, we will have the last row, which will be, so we will start with 72Z, and then we will filter it by the, so we will take 72Z, intersect 12Z, and that contains 72Z intersect 36Z. So, that is the series we get.

And now simplifying it, we get Z contains, and here Z intersect 12Z is 12Z, 12Z plus 18Z, so we have to look at the GCD of 12 and 18, which I believe is 6 again. And then, here we have the 36Z, 36 and 18 GCD is 18. Now, it is different from the previous series we had, and 18 intersect 12. So, that is the LCM of 18 and 12, that is 36 plus 72. So, that is still 36Z. And then, here again, we have 18 intersect 36, that is 36Z plus 72Z, that is still 36Z, so that is the same. And then, we have 72Z, and you do not really, these are now both 72Z. So, the final step is 0.

Let us just check that. So, this refines 2. So, let us just check that these are indeed equivalent by writing down the quotients. So, I just write on the order of the quotient in each case. So, that is because just going to be a cyclic group. So, here it is 6, and then this is 2 and then, so 6 Z mod 12Z, this is a group of order 2, 12Z mod 36Z is a group of order 3, 36Z mod 72Z is again a group of order 2. So, the quotients here are Z mod 6Z, Z mod 2Z, Z mod 3Z, Z mod 2Z, the questions are Z mod 6Z, Z mod 3Z, Z mod 2Z, Z mod 2Z. So, this is just a slight permutation here, these 2 and 3 get interchanged.

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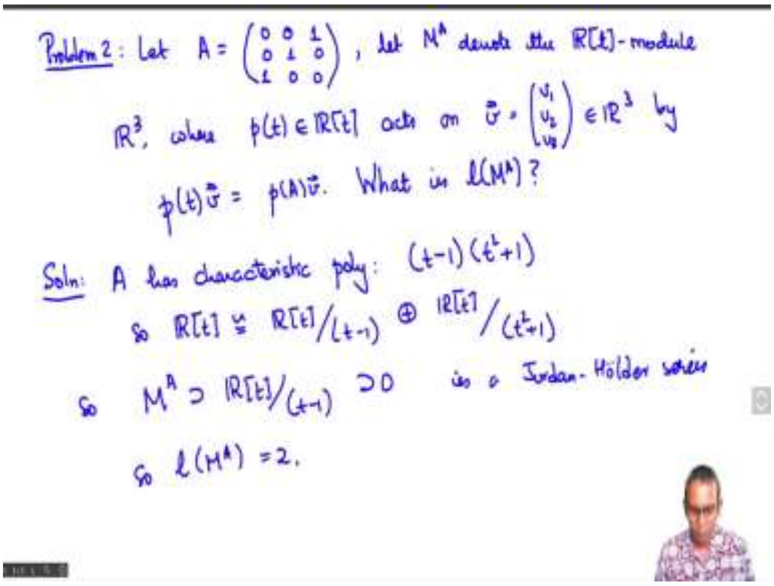

Now, let us come to problem 2. Let A be the matrix 0, 0, 1, 0, 1, 0, 1, 0, 0. And let M superscript A denote the R t module with the real numbers R cubed, where p t a polynomial in R t acts on v, which I will write as a column vector v1, v2, v3 belongs to R cubed by p t, v equals p of A. So, this is the usual way in which we attach R t modules to matrices with increase in R. So, the problem is, what is the length of M A?

So, just take a minute, pause the video take a minute and try to solve it yourself. Otherwise, let me solve it for you. So, you see A has characteristic polynomial, I guess this t minus 1 it has one eigenvalue 1 and then it has two eigenvalues which are not real plus or minus i. So, it has characteristic polynomial t minus 1 times t square plus 1, and these are both irreducible. So, this is a factorization of the characteristic polynomial into irreducible factors.

And so, R t is isomorphic to R t mod t minus 1 plus R t mod t square plus 1. And both these are simple because these polynomials are irreducible over R. So, if we take M A contains R t, you could take either one of them, is a Jordan-Holder series. And so, length of M A is equal to 2. And a similar problem, in the same way, let us solve.

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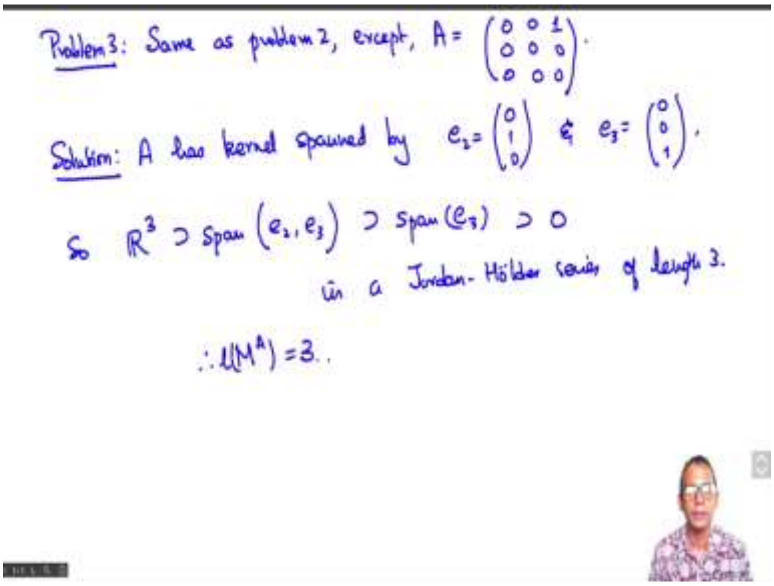

So, problem 3 is the same as problem 2, except we will change the matrix A. And let us take A to be this matrix just a 1 in the corner, and 0's everywhere else. So, pause the video and try to solve it. Here is the solution, so A has kernel spanned by two vectors, e2 and e3. So, e2 is the vector, maybe I will just write it down, 0, 1, 0 and e3 equals 0, 0, 1, this is easy to check. And so, R cubed contains span of e2, e3 contains a span of e3 contain 0, this is certainly a composition series because the kernel is invariant.

And of course, also in subspace of the kernel is invariant for a linear operator. And these portions are 1-dimensional over R. So, they have to be simple because every submodule of an R t module is going to be an R subspace. So, this is a Jordan-Holder series. And so, since this is M0, M1, M2, M3 this is has length 3. Therefore, M A has length equal to 3.