Algebra – II
Professor. Amritanshu Prasad
Department of Mathematics
The Institute of Mathematical Sciences
The Jordan-Holder Theorem

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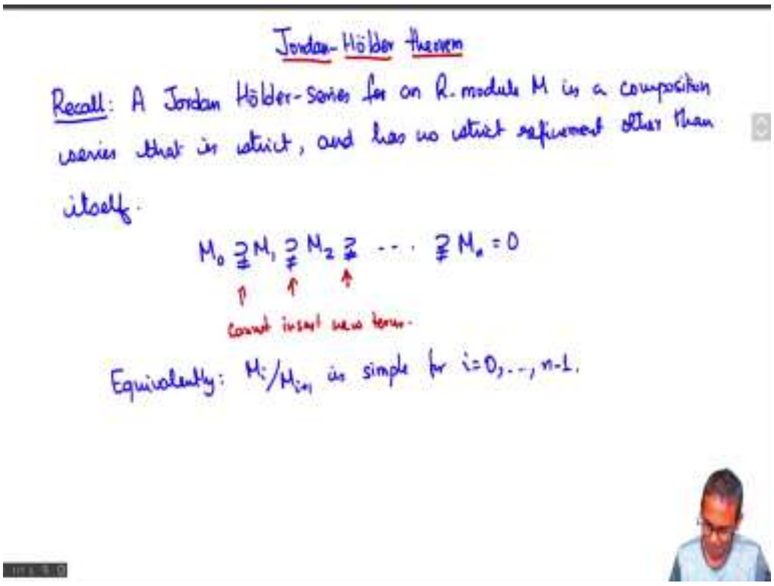

In this lecture, I am going to discuss the Jordan-Holder theorem. Recall that Jordan-Holder a series for an R-module M is composition series, that is strict and has no strict refinement other than itself. So, a Jordan-Holder a series looks something like M0 contains M1 contains M2. So, each submodule properly contains the next one, the last one is 1, and there is no way to insert, cannot insert new terms in between.

And more importantly, equivalently, we could say that Mi mod Mi plus 1 is simple for i equals 0 to n minus 1. Now, we have seen some conditions, namely the ascending chain condition and the descending chain condition which ensure the existence of a Jordan-Holder series. But let us not worry about that right now. I will state the Jordan-Holder theorem in more general form, and it turns out to be a direct consequence of Schreier’s theorem.

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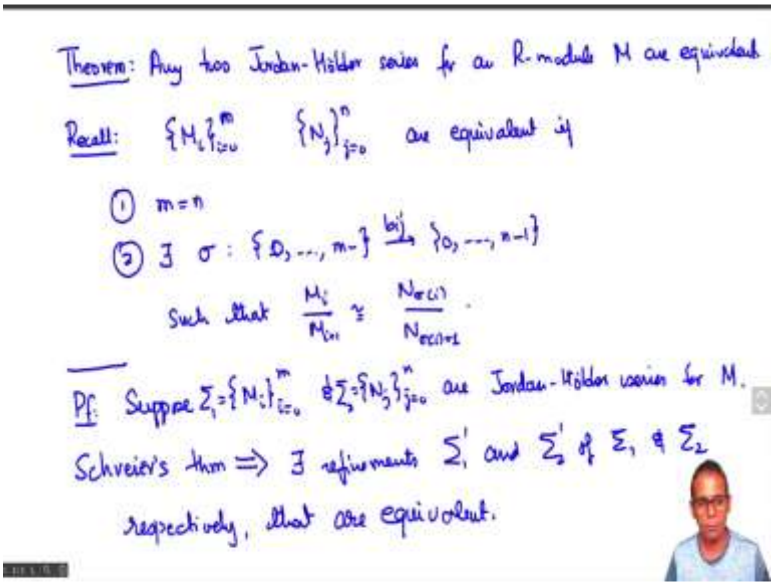

So, let us state the theorem. A Jordan-Holder, if let us just, yeah, I think the simplest statement would be any two Jordan-Holder series for an R-module M are equivalent. So, let us just decode what this is. Firstly, I am not claiming that an R-module has a Jordan-Holder series, I am just saying that if it has two Jordan-Holder series, they must be equivalent. So, what this means.

Firstly, recall what this means. This means that if you have Mi, i equals 0 to n, Nj, j equals 0 let us say i equals 0 to m and j equals 0 to n are equivalent. If 1, M is equal to n, and 2, the exist sigma from the set 0 to m minus 1 to 0 to n minus 1 of bijection, which kind of implies that M is equal to N such that Mi modulo Mi plus 1 is isomorphic to N sigma i modulo N sigma i plus 1. And this will follow directly from Schreier's theorem.

So, proof, suppose, Mi, i equals 0 to m and Nj, j equals 0 to n are Jordan-Holder series. So, then Schreier's theorem says that, so let us call this sigma 1 and let us call the sigma 2. There exist the refinements sigma 1 prime and sigma 2 prime of sigma 1 and sigma 2 respectively, that are equivalent. But now, sigma 1 and sigma 2 are already Jordan-Holder series, they cannot be refined by strict series. So, the sigma 1 prime and sigma 2 prime will just be refinements of sigma 1 and sigma 2, where you are adding more terms which are equal to the existing terms in the series.

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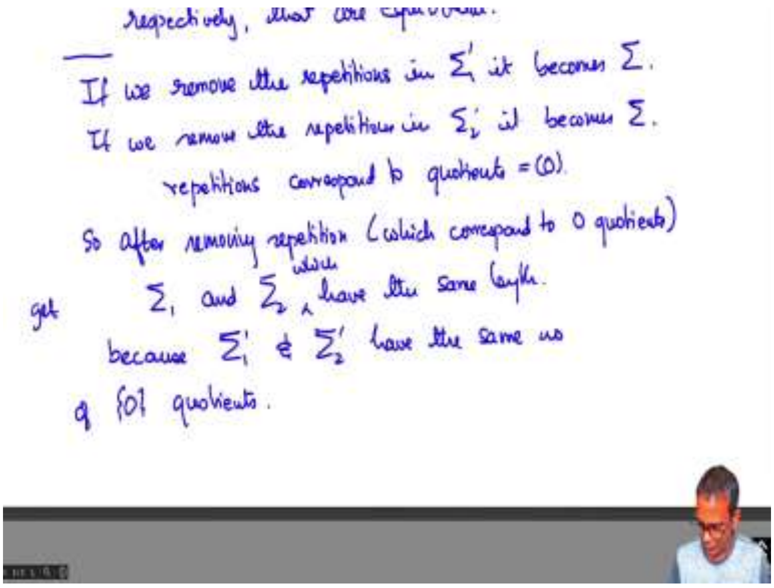

So, what we are saying is that, if you have sigma 1 is something like this M0, M1, M2, M n equals 0, then you will be adding terms in between which are either equal to, so you may be adding terms like this M0 1, M0 2 or many more terms are in between these two terms will be either equal to M0 or to M1. So, there are no new terms that appear in sigma 1 prime and sigma 2 prime.

And so, once you take, so if you take, so if we remove the repetitions in sigma 1 prime it becomes sigma and if we remove the repetitions in sigma 2 prime it becomes sigma. But these repetitions are precisely those terms in the series where the quotients are 0's and because sigma 1 prime and sigma 2 prime are equivalent, they have the same number of 0 quotients.

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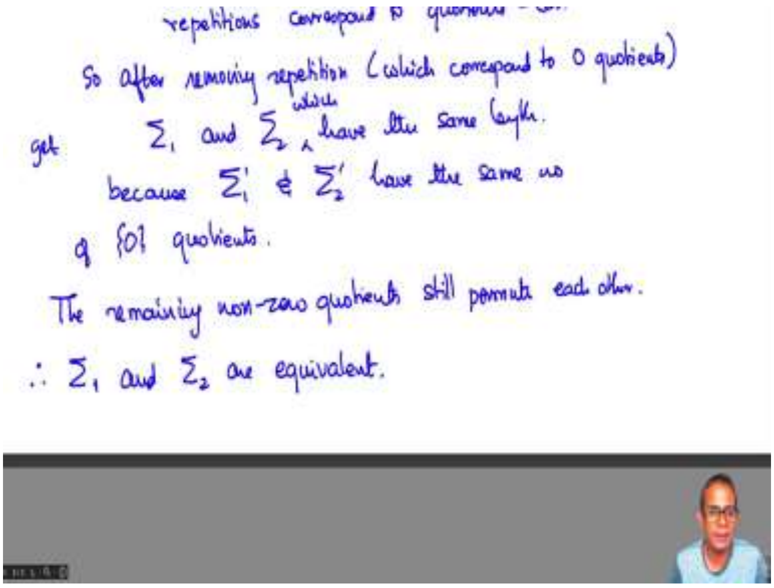

So, after removing repetitions which correspond to 0 quotients sigma 1 prime and sigma 2 prime have the same length. So, since they have the same number of, well sigma 1 prime and sigma 2 prime have the same length. So, sigma 1 and sigma 2 have the same length. So, you get sigma 1 and sigma 2 which have the same length because sigma 1 prime and sigma 2 prime have the same number of 0 quotients. And the remaining quotients of course, they permute each other.

And the remaining non-zero quotients still permute each other. So, we have a bijection between the 0 quotients, after we remove them, we will still have a bijection between the non-zero quotients. And so, these two Jordan-Holder a series sigma 1 and sigma 2 have to be equivalent. So, we have not assumed here that M has satisfies the ascending chain condition or the descending chain condition, we just assumed that M has Jordan-Holder series.

We have not assumed that M satisfies the ascending chain condition or the descending chain condition. But just with the assumption that M admits at least one Jordan-Holder series, we can do.

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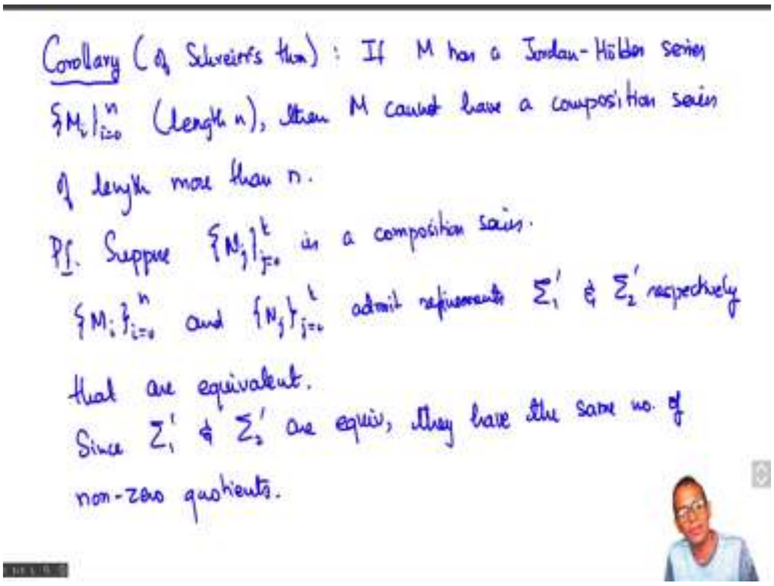

So, here is a corollary, and there is not a corollary of the Jordan-Holder theorem, as stated, per se, it is a corollary of Schreier's theorem, which says that if M has a Jordan-Holder a series of length l. And so let us say Mi, i goes from 0 to n. So, that means length, I call that length n, then M cannot have a strict composition series of length more than n. So, every composition series of M will have length less than or equal to n.

And the proof is this, suppose, Nj j goes froM0 to k is a composition series then Nj and Mi, so let us call this. So, so, Mj i equals 0 to n and Nj j goes from 0 to k admit the refinements sigma 1 prime and sigma 2 prime respectively. So, sigma 1 prime is a refinement of this Jordan-Holder series Mi and sigma 2 prime is a refinement of this composition series Nj that are equivalent.

Since they are equivalent, they have the same number of non-zero quotients. Moreover, the number of non-zero quotients of a refinement would be at least as large as the number of non-zero quotients of a composition series.

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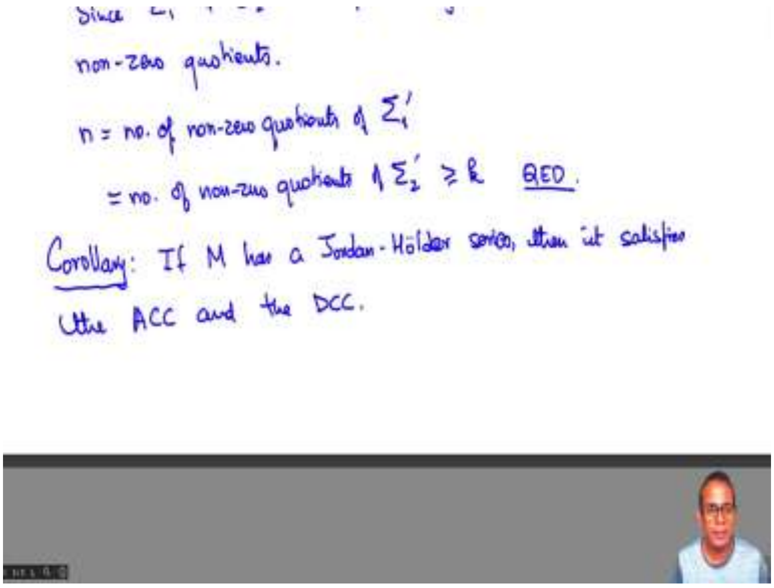

So, now what we have is n is the length of the composition series Mi. So, it is the number of non-zero quotients of sigma 1 prime and that is equal to the number of non-zero quotients of sigma 2 prime. But that has to be greater than or equal to k because a sigma 2 prime is a refinement of Nj and so k has to be less than or equal to m. So, you cannot have any strict composition series of length greater than the length of a composition series, of a Jordan-Holder series.

Corollary: If M has a Jordan-Holder series then it satisfies the ACC and the DCC. Because if you have an increasing chain, it has to stabilizes at some point because you cannot have a strict increasing chain of length more than the length of the Jordan-Holder series, and the same applies for decreasing chains. And also, the Jordan-Holder theorem gives us a set of invariants for a module.

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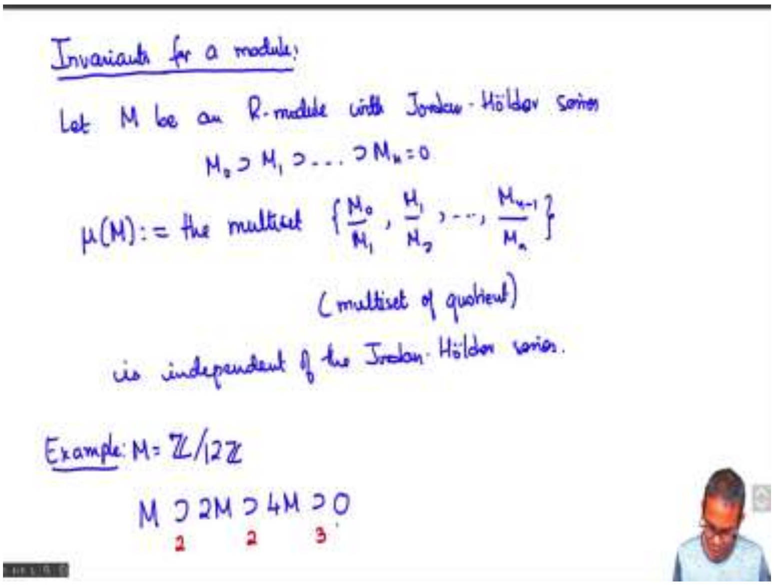

So, here is how we compute this. So, let M be an R-module bit Jordan-Holder series, M0 contains M1 contains Mn equals 0. So, this is going to be strict as the quotients are going to be simple, the simple modules. So, we will call this mu of M, define mu of M to be the multiset M0 mod M1, M1 mod M2, Mn minus 1 mod Mn. So, we remember the multiplicities with which the different quotients arise. So, this is the multiset of quotients.

This is independent of the Jordan-Holder series, this is what equivalence means. For example, we had looked inside Z mod 12 Z. So, let us call this M, this is the Z module and this had submodules for example, we have a Jordan-Holder series which is M well contains 2M contains 4M contains 0, and the quotients here are Z mod 2, Z mod 2, and Z mod 3.

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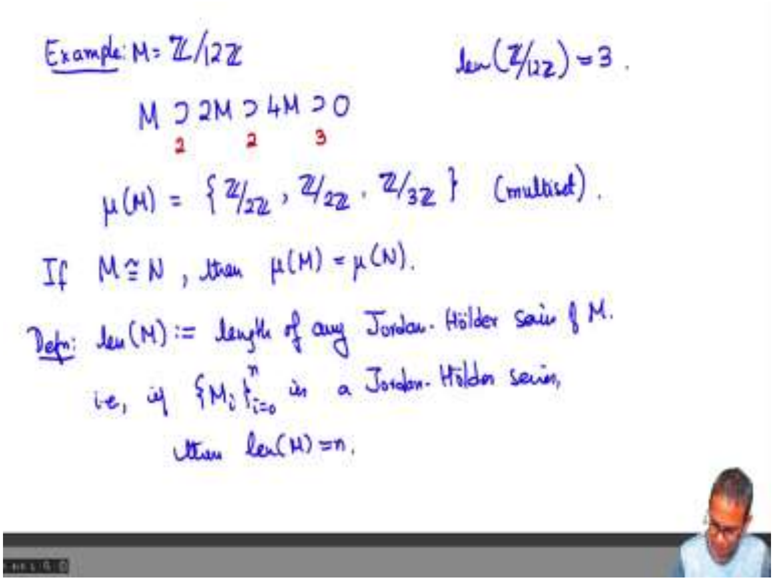

So, mu of M is, mu Z mod 2 Z, Z mod 2 Z, Z mod 3 Z, a multiset. This does not depend on the composition series. And if you have two isomorphic modules, then these multisets are equal. The size of this multiset is called the length of the module. So, definition, length of M is defined to be the length of any Jordan-Holder series of M. Just to be doubly clear, if Mi goes from M0 to n is a Jordan-Holder series, then we will say that length of M is equal to n. So, what we have in this example is that the length of Z mod 12 Z is equal to 3.