Algebra – II
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Existence of Jordan-Holder Series

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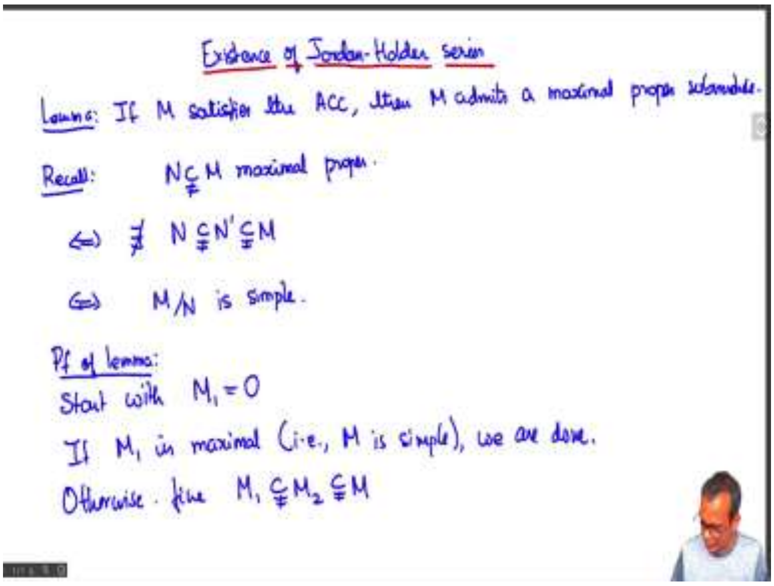

I will now show that if an R-module M satisfies the ascending chain condition and the descending chain condition, then it admits a Jordan-Holder series. So, to prove that first we prove the following lemma that if M satisfies the ascending chain condition, then M admits a maximal proper submodule.

So, recall that a maximal proper submodule. So, firstly it means that it is proper and secondly, it means that there does not exist any N prime between N and M. But this is the same as saying that M mod N is simple because the submodules of M that contain N are in bisection with submodules of M mod N. So, now let us prove this. So, what we will do is we will start with M1 equal to 0.

Now, we lost if M1 is maximum which means that M is simple, then we are done. Otherwise, find M1, M1 is strictly contained in M2 which is strictly contained in M.

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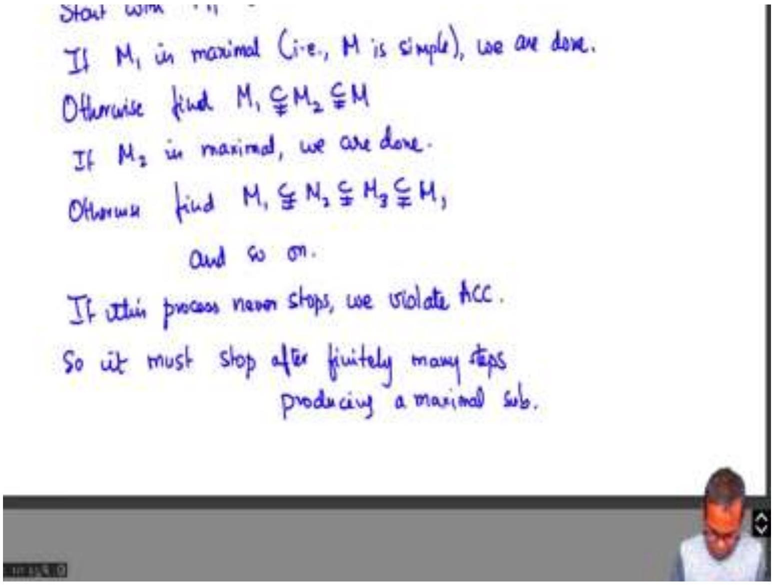

And now, either M2 is maximal we are done. Otherwise, we can find M1 strictly contained in M2 strictly contained in M3 strictly contained in M and so on. This process if it never stops, we will have a contradiction to the ascending chain condition. And so, it must stop at after finitely many stages producing a maximal submodule. So, this is how we prove that every module which satisfies the ascending chain condition has a maximum proper submodule. Now, the next stage we can construct Jordan-Holder series. So, now we go to the construction of a Jordan-Holder series.

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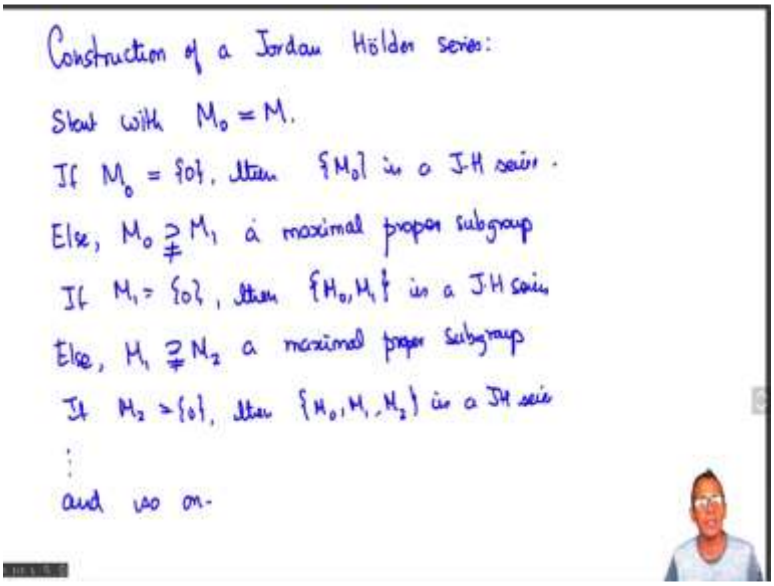

So, you start with M0 equal to M. If M is, so now at each stage what to do is if M0 is equal is trivial then there is nothing to do, then just the singleton M0 is a Jordan-Holder series, there are no quotients here. Else, M0 also satisfies M0 contains M1 a maximal proper subgroup. So, M0 mod M1 is simple. If M1 equals 0, then M0, M1 is a Jordan-Holder series.

Otherwise, well note that M1 must also satisfy the ascending chain condition, it inherits the setting chain condition from M0 because every sequence of modules, submodules of M1 is also a sequence of submodules of M0. So, M1 again satisfies the ascending chain condition. So, M1 contains M2 a maximal proper subgroup. If M2 is trivial, then M0, M1, M2 is a Jordan-Holder series just because each of these questions is simple and so on. So, now, once again as we argued before, if this process never stops, then we will have a violation of the descending chain condition.

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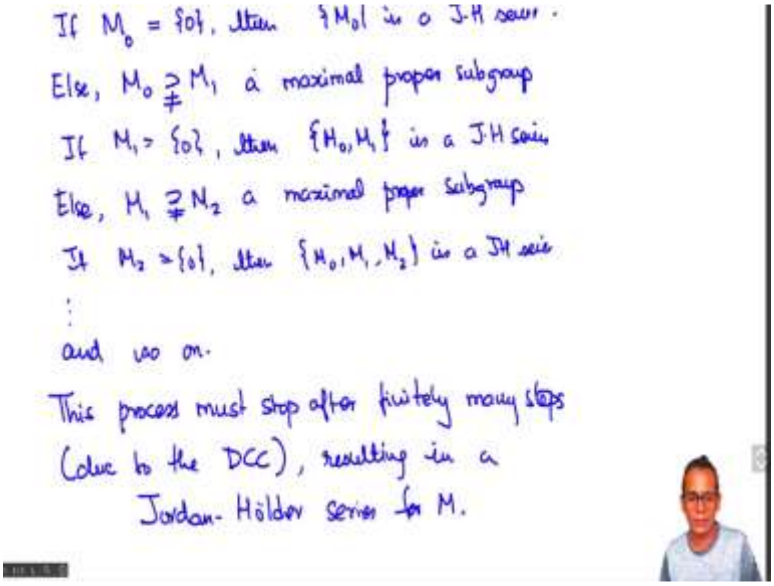

So, this process must stop after finitely many steps, due to the descending chain condition and when it stops you have a Jordan-Holder series. Resulting in a Jordan-Holder series for M. In some sense, both these proofs are algorithmic. The first one tells you how to construct a maximal submodule and the second one tells you how to construct the Jordan-Holder series by taking a sequence of maximal proper submodules.