Algebra – II
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Ascending and Descending Chain Conditions

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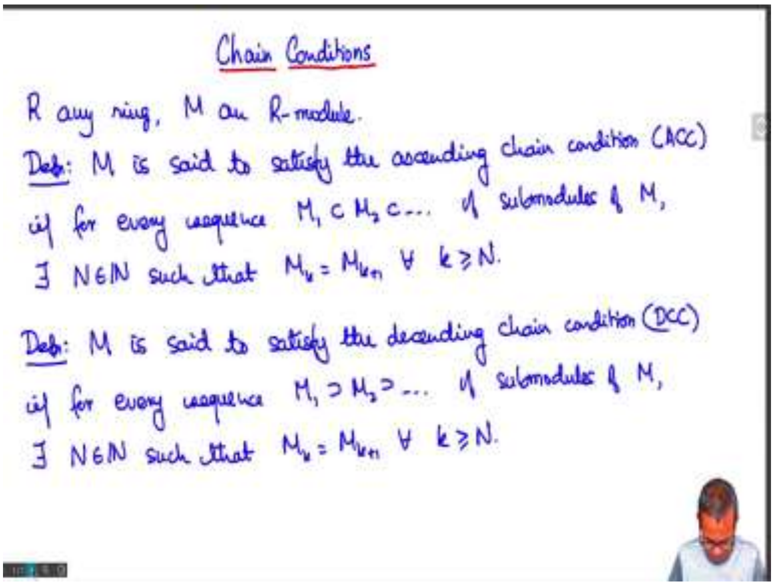

In this lecture, I will introduce chain conditions. These are certain conditions which taken together, will ensure that a module has a Jordan-Holder series. So, let R be any ring, M and R-module. And now, we will define the ascending chain condition, we say that M is said to satisfy the ascending chain condition, which is also known as the ACC. If for every ascending chain, if every ascending chain of submodules of M stabilizes

So, if for every sequence M1 contained in M2 contained in so on of submodules of M, there exists natural number N such that Mk is equal to Mk plus 1 for all k greater than or equal to N or in other words Mk is equal to Mn for all k greater than or equal to N. And similarly, we have the descending chain condition which is defined in a very analogous manner.

So, we will define the DCC or descending chain condition and here we just reverse the containment. These are two conditions, together they will ensure that, we will see that they will ensure that the module M will have a Jordan-Holder series. Let us look at some examples.

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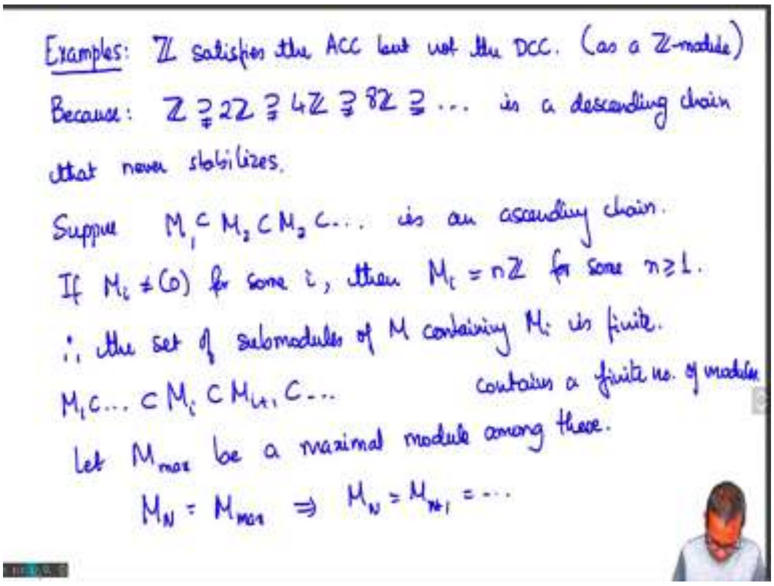

Z satisfies the ascending chain condition but not the descending chain condition. Firstly, let us show that Z does not satisfy the descending chain condition. So, I just need to find an infinite descending chain of submodules. So, here we think of Z as a Z module or this is the same as looking at abelian group. So, here is a descending chain, you just take Z strictly contains 2Z which strictly contains 4Z which strictly contains 8Z and so on. This is a descending chain which never stabilizes.

And the other way, suppose we have M1 contained in M2 contained in M3 is an ascending chain. Suppose, at least one of these is not 0. So, if Mi is not equal to the 0 model for some i, if Mi 0 for all i, then there is nothing to prove because then it stabilizes right from the first stage. So, if Mi is non-zero for some i, then what you can do is, you can Mi is of the form n Z for some n greater than or equal to 1.

So, Mi is going to be of this form because this is what all subgroups of Z and therefore, all the submodules of Z look like. So, therefore, the set of submodules of M containing Mi which is the same as which is going to be in bijective correspondence with submodules of Z mod n Z is finite. Therefore, when you look at the modules Mi, Mi plus 1 this contains a finite number of modules, only a finite number of modules occur in this sequence.

And so, even if you take this starting from M1, the entire sequence only a finite number of modules appear in it. So, you can take the largest of such modules. Let M max be the largest such module, or rather I should say a maximal module of this kind, may not be unique. And, well it has to be unique because of the chain. So, let M max be a maximal module among these.

And if Mn is equal to M max, then this implies that Mn is equal to Mn plus 1, and so on. The rest of the modules have all to be equal to this. And so that shows that the integers satisfies the ascending chain condition, but not the descending chain condition. And now let us look at an example of Z module that satisfies the descending chain condition but not the ascending chain condition.

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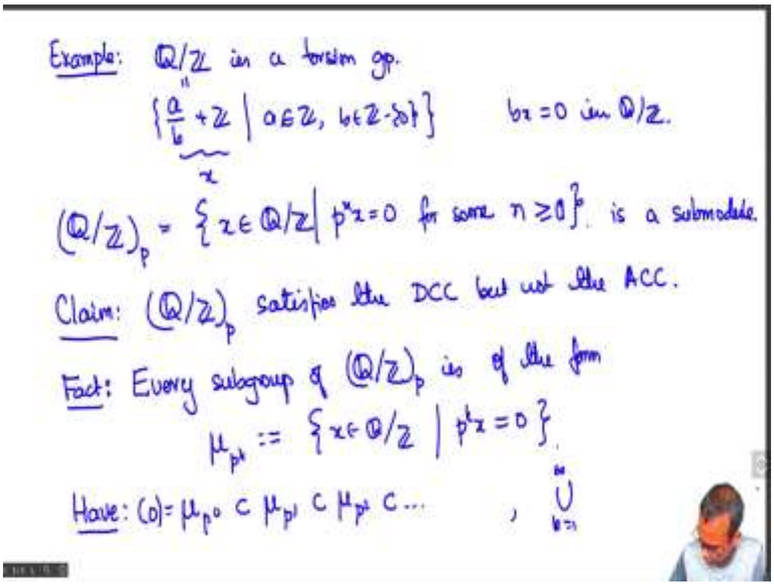

So, to construct this example, let us first look at the group Q mod Z. So, Q mod Z is a torsion group. In the sense that every element of Q mod Z is a torsion element that is an element here is of the form a over b plus Z, it is a co set of a over b plus Z where a belongs to Z, and b belongs to Z minus 0. And if you take this element and multiplied by b, so if I call this x, then bx is equal to 0 in Q mod Z because bx is going to be an integer and get subsumed in this Z.

Now, we will define this p primary part Q mod Z p is defined to be x belongs to Q mod Z such that p to the n x equals 0 for some n greater than or equal to 1 or we could say some n greater than or equal to 0. So, that is a subgroup or a submodule if you want to think of them as Z modules, I claim that Q mod Z p satisfies the DCC, descending chain condition, but not the ascending chain condition.

Well, to prove this, you use the following fact which I will leave as an exercise for you to do. Every subgroup of Q mod Z p is of the form mu p k which is defined to be x belongs to Q mod Z such that p to the kx equals 0. And so, these subgroups themselves they form a chain. So, we have a new p to the 0 which is just a subgroup consisting of 0 alone. You are seeing x is equal to 0 then this contains mu p 1, this contains mu p 2 and so on.

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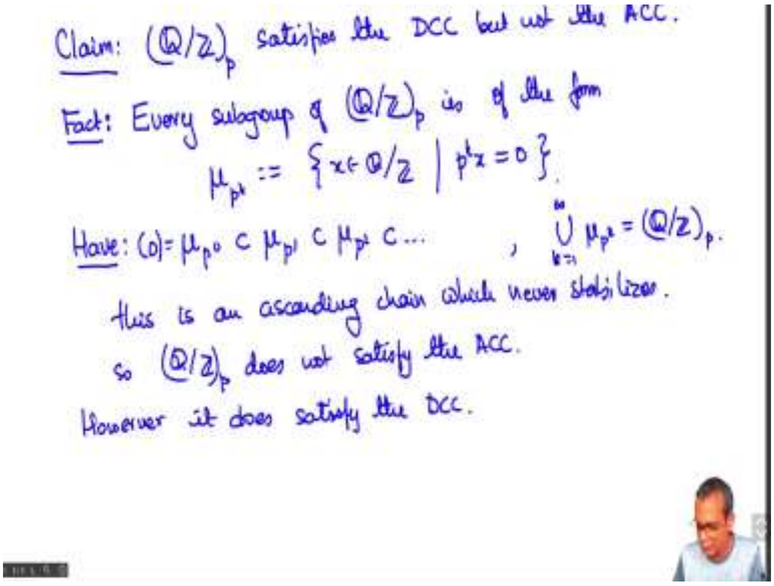

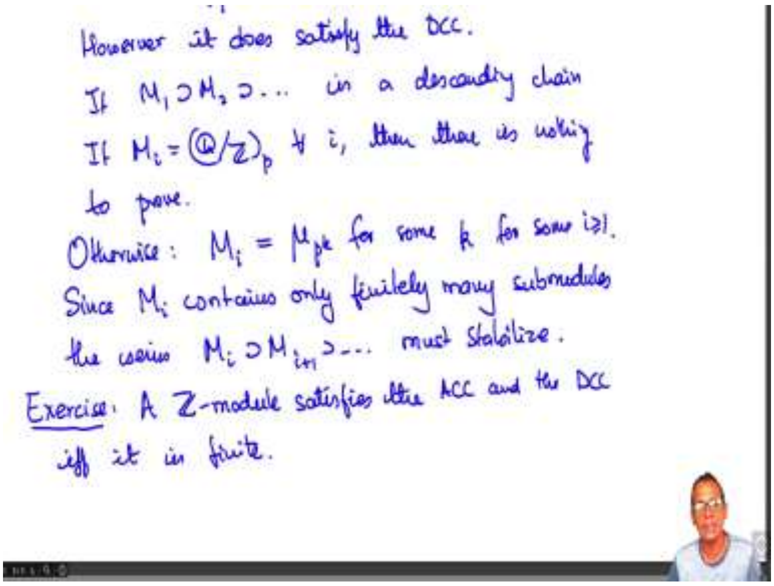

And the union of these things is Q mod Z. So, we have only all the sub groups of this group Q mod Z subscript p from a single chain. And so, if you have any chain here, it will just take terms from this. Now, if you start, so clearly, if you have a descending chain, it will terminate because when you start if it has any of these elements, you have only finite many subgroups contained in it.

Whereas, this chain itself is an ascending chain. So, this is an ascending chain which never stabilizes. So, Q mod Z p does not satisfy the ACC. However, it does satisfy the DCC and for this we will argue as in the case of integers. Suppose, you have it descending chain. Well, it could be that all the entries of this chain are the full group, full module itself, if Mi equals Q mod Z p for all i, then there is nothing to prove.

Otherwise, we have Mi equals mu p to the k for some k for some i. So, we have at least one term in this series, which is a proper subgroup of Q mod Z p and then after the only finitely many submodules the series Mi contains Mi plus 1 must stabilize. Now, you could ask what are the Z modules which satisfy both the ascending chain condition and the descending chain condition.

And here is an exercise for you, Z-module satisfies the ACC and the DCC if and only if it is finite. Clearly, if it is finite, then it can have only finitely many submodules and so it must satisfy the ACC and the DCC. On the converse, you need to think about why it is true. So, if I Z-module is infinite, then either the ACC or the DCC must fail. I will leave you with that. And in the next lecture, we will see how if you have a module that satisfies both the ACC and the DCC, then it admits a Jordan-Holder series.