Algebra – II
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Lecture 73
Schreier's Theorem

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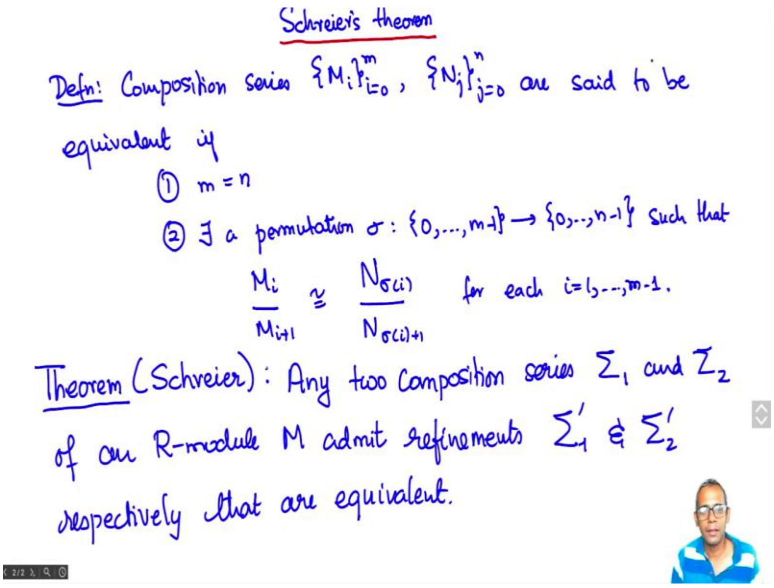

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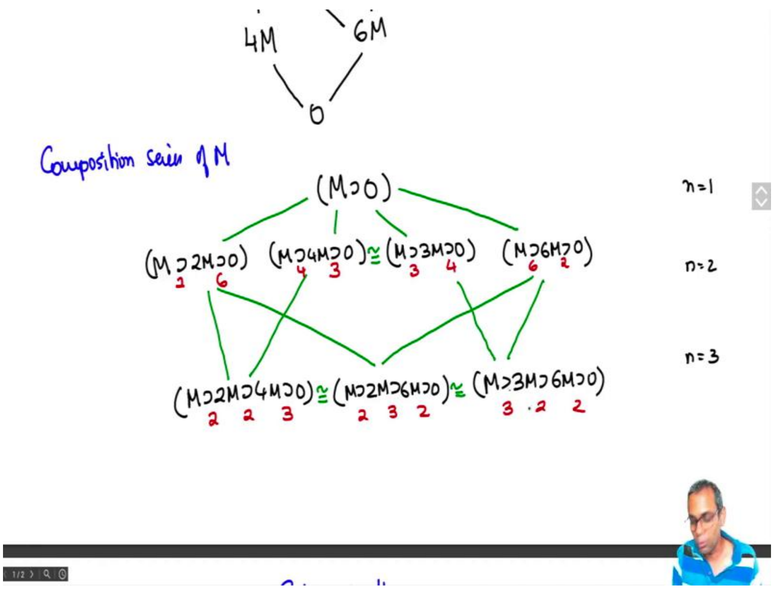

Our objective is to prove the Jordan-Holder theorem, and the path that we will take towards the Jordan-Holder theorem is via a theorem called Schreier's theorem. Which is a very general theorem with no assumptions on the ring R, whose modules we are considering. So, let us look at Schreier's theorem. So, firstly, before we do that, I need to make one more definition about the equivalence of two composition series. Two composition series, let us see, the first one is M i, i goes from 0 to m.

And the second one, N j, j goes from 0 to n are said to be equivalent, basically if their quotients are the same. More precisely, if, firstly, m is equal to n and secondly, there exists a permutation, let us call it sigma from the set 0 to m minus 1 to the set 0 to n minus 1 such that the quotient corresponding to i let us say, M i mod M i plus 1 is isomorphic to the quotient corresponding to sigma i in the second composition series.

So, n sigma i mod, N sigma i plus 1, for each i in 1 to in the range of 1 to m minus 1. So, two composition series are equivalent if their quotients are the same. So, last time we had looked at an example of composition series, which started with the ring R to be Z and we looked at the R-module M equals Z mod 12 Z and we had listed firstly all the sub-modules of this module and then using that we had also listed composition series.

And in red here I have also the quotients listed, the integer i means I am looking at Z mod i Z as the quotient. So, there was a small mistake here, this 2 and 6 need to be interchanged. I do not know if you noticed it, this is 6 and this is 2. So, now, if you see all these three things here in the last these are equivalent, this is equivalent this, this is equivalent this because there are 3 quotients, 2 of them are Z mod 2 and one of them is Z mod 3.

So, you can permute them, permute these things and get these, but over here, what we have is that this is equivalent to this. And these two, this composition series is equivalent to this composition series. So, we have two equivalence classes for composition series with n equals 2 and we have a single equivalence class with composition series with n equals 3.

In fact, this last thing that all the composition series of longest length, which turn out to be Jordan-Holder series are all equivalent is the Jordan-Holder theorem, but in any case, what you see here is that given any two series (comp) composition series, there is for example, you could take 2 comma 6 and for these two series, this is a common refinement.

Whereas, if you take this series and this series, the first and third in the second row that is M contains 2 M contains 0 and M contains 3 M contains 0, they do not have a common refinement. However, they both have refinements, which are equivalent. And that is basically the content of Schreier’s theorem. So, let us state Schreier’s theorem.

Any two-composition series sigma 1 and sigma 2 of an R-module M, admit refinements, let us call them sigma 1 prime and sigma 2 prime. That means sigma 1 prime is a refinement of sigma 1 and sigma 2 prime is a refinement of sigma 2. So, let us write respectively, such that

that are equivalent. And we saw how this works in the example of Z mod 12. The proof of Schreier's theorem rests upon a very simple idea.

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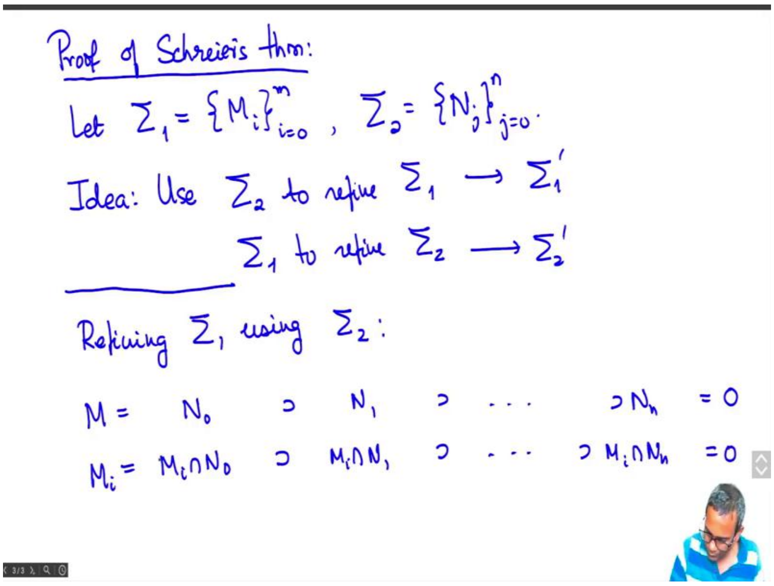

So, let us get, let me explain the idea to you. So, firstly, let us give these composition series name, so we have, let us say sigma 1 is just the series M i, i goes from 0 to m. And let us say sigma 2 is the series N j, j goes from 0 to n. Now, the idea behind the proof is, use sigma 2 to refine sigma 1 and use sigma 1 to refine sigma 2. So, these will give rise to refinements, sigma 1 prime, and sigma 2 prime.

And we will show that these refinements are actually equivalent. So, how do we do this? So, let us start with firstly refining sigma 1 using sigma 2. So, let us start with the sigma 2 itself. So, sigma 2 is this thing. So, you start n 0 is M. And then that contains N 1 that contains and so on, all the way down to N n, which is equal to 0.

Now let us take this and intersected with M i, so I get M i is equal to M i intersect N 0, which contains M i intersect N 1 all the way to M i intersect N n, which is 0. So, this is not what I want, I do not want to go refine, I want to refine the sigma 1, which is the M. So, I do not want to end up at 0 here, I want to end up at M i plus 1. So, what I do is, I add M i plus 1 to each of these things.

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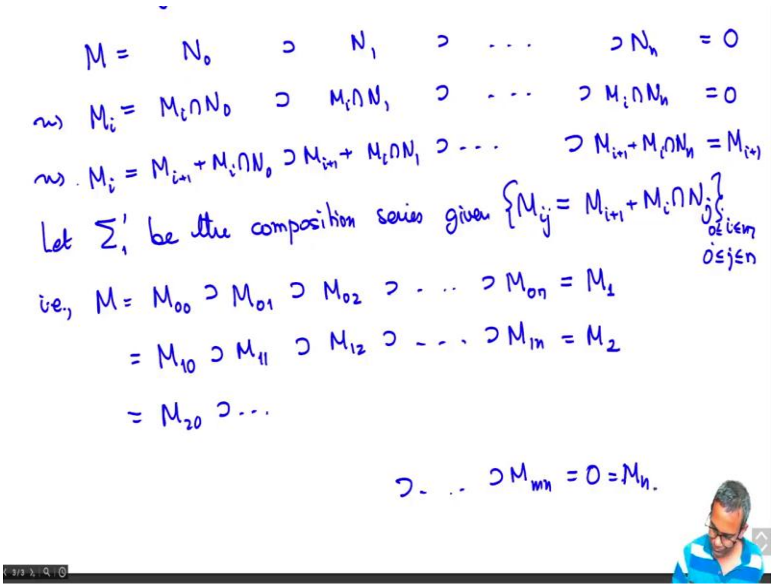

So, the next step is we have M i, but let us just add M i plus 1 to all these modules. So, this is the same as M i plus 1 plus M i intersect N 0. And that contains M i plus 1 plus M i intersect N 1, and so on. And finally, we have M i plus 1 plus M i intersect N n. Can this last term is equal to M i plus 1?

Now, in this refinement, it is quite possible. So, this is actually a refinement of M let me write it down more explicitly. But it is quite possible that many of these terms here are going to coincide with each other. There will be no change between this module here and the next module, it is quite possible that will often happen.

So, we are not talking about strict composition series, here we just talk about composition series, where terms are allowed to repeat. So, finally, what we get is the composition series sigma 1 prime. So, sigma 1 prime, let us see, let sigma 1 prime be the composition series. So, given by M i j equal to M i plus 1, plus M i intersect N j. So, what we write is, so, first we start with M equal to M 0 0.

And that contains M 0 1, that contains M 0 2. So, here now j is changing as we had in each row of this here. So, j is changing, j is changing as we had in each row over here. And this goes all the way up to M 0 n and this is equal to M 1. And so, that is again, we go on out to the next terms, which is, so M 1 is still equal to M 1 0, which contains M 1 1, which contains M 1 2, and so on, which goes down to M 1 n, which is now equal to M 2. And this is equal to M 2 0 and so on.

And finally, we will end up with all the way down to M m n, which is 0. So, the sigma 1 is this composition series given by this and we have two indices, we take i goes from 1 to n and j goes from 1 to m, 0 to n, i goes from 0 to m, and j goes from 0 to n. First, running the index is j, within each index i. So, this is I hope, it is clear what sigma 1 prime is, and this is clearly a refinement of sigma 1 because I have the terms of sigma 1 in it, these things M 1, M 2, and this is M n. Same in a similar manner, we will refine sigma 2 using sigma 1.

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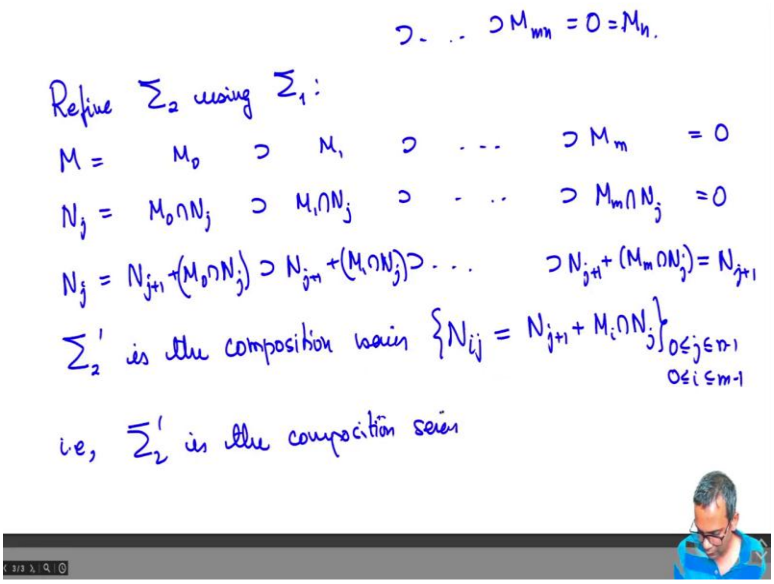

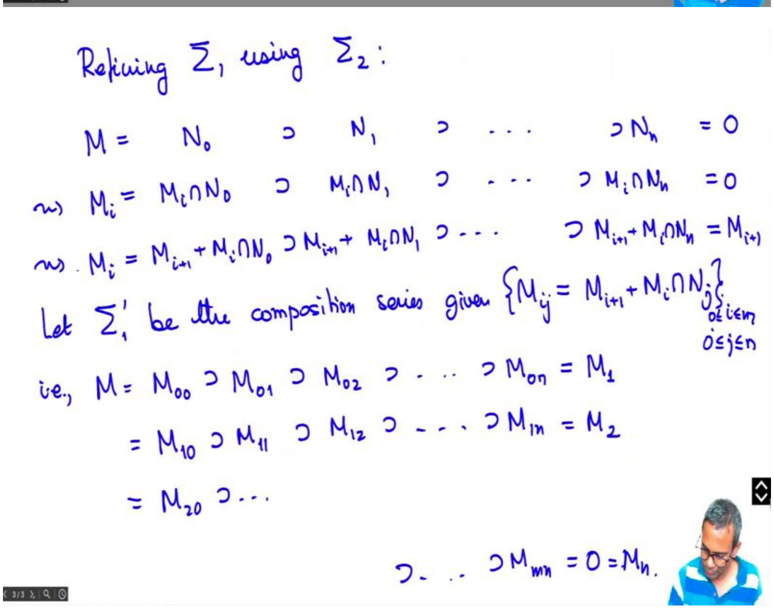

So, now we will refine sigma 2 using sigma 1. Let me just do it again just to give you the hang of it. So, what we are starting with is the filtration sigma 1, so that is given by M equals M 0 contains M 1 contains all the way up to M m, which is equal to 0. And then what we will do is we will intersect it with N j.

So, what we get is N j equals M 0 intersect N j contains M 1 intersect N j contains M m intersect N j and this is still 0 and now let us add N j plus 1 to each term of this series. So, we get N j equals N j plus 1 plus M 0 intersect N j, this contains N j plus 1 plus M 1 intersect N j and so on. And here we have N j plus 1 plus M m intersect N j and this is equal to N j plus 1.

So, what we have is a new composition series, which we will call sigma 2 prime. So, sigma 2 prime just like we have sigma 1 prime it is a composition series given by, we will call it N i j equal to N j plus 1 plus M i intersect N j. And now we first run over the index ci and within that we run over the index i, so 0 less than or equal to j less than or equal to n minus 1, 0 less than or equal to i less than or equal to M minus 1. More explicitly, what we are looking at is the composition series.

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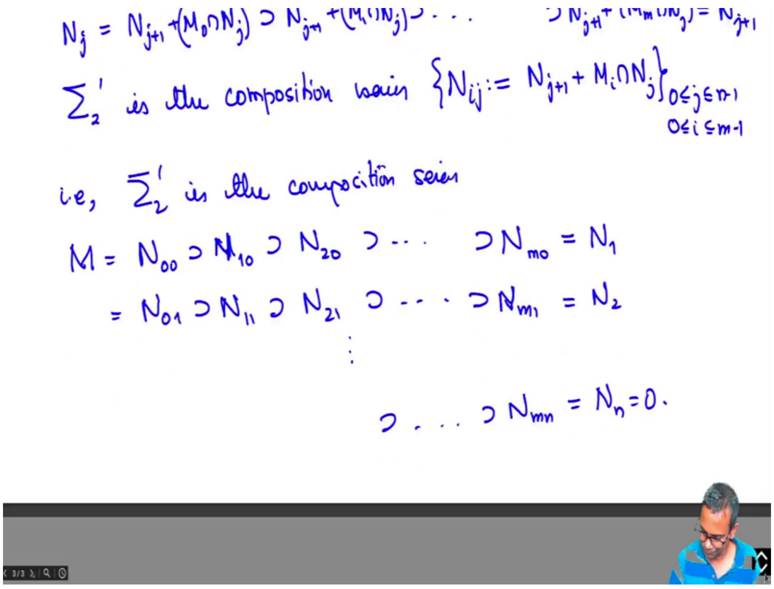

So, we start with M, which is equal to N 0 0, which contains N 1 0, which contains N 2 0. So, the order of the indexing here is reversed. And this is equal to N 1. And then this is the same as N 0 1, which contains N 1 1 which contains N 2 1, which contains, this should be an N m 1, this is N 2, and so on. All the way down to N m n, which is N n, which is 0. So, this is a refinement of the composition series, sigma 2. Now let us see what the quotients of these two series are.

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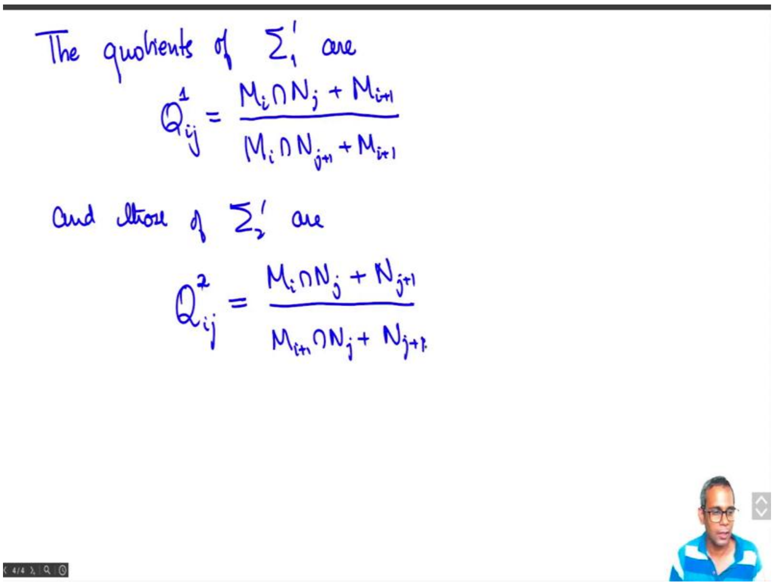

So, the quotients of sigma 1 prime, well, there are some 0s, but the remaining are of the form Q i j equal to M i intersect N j plus M i plus 1, that is the i jth term of the series sigma 1 prime, and then we must increase j by 1 because in this the variable j, the index j is running inside the index i.

And So, we have M i intersect, and j plus 1 plus M i and those of sigma 2 prime are Q prime i j equal to or maybe I will call this Q 1 and I will call this Q 2 that makes more sense and these are M i intersect N j plus N j plus 1 modulo M i plus 1 intersect N j plus N j plus 1. So, these are the quotients of the two refinements that we have constructed, the situation looks something like this. So, there are 4 relevant R-modules in this.

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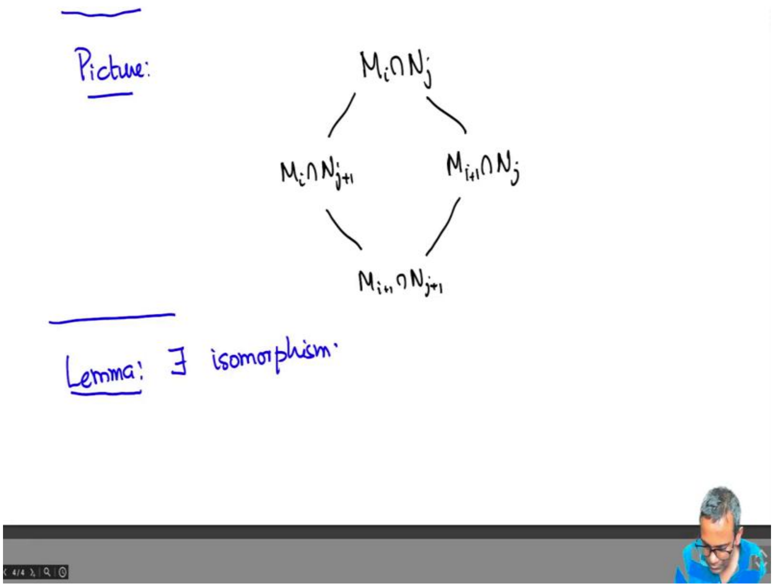

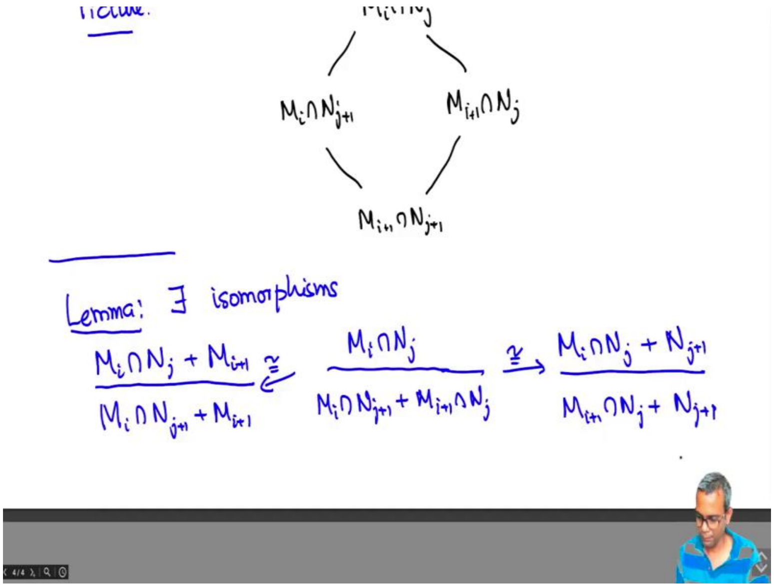

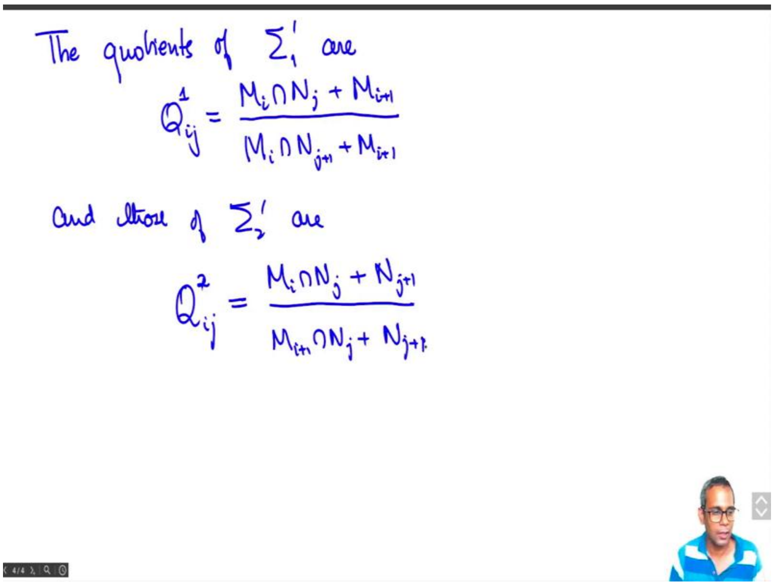

So, just a schematic picture you have, M i intersect N j this contains two sub-modules M i intersect and j plus 1 and M i plus 1 intersect N j and then we have here the smallest module which is contained in all these which is M i plus 1 intersect N j plus 1.

Now it turns out that both these quotients are coming from this picture what they capture is you take this module M i intersect N j and you go modulo everything that comes from here. So, let me just make that statement. So, lemma is that, there exist isomorphisms which I will construct in the proof.

So, we have this stuff on top M i intersection N j modulo everything that is below it which means I must take M i intersect N j plus 1 plus M i plus 1 intersect N j, this quotient is isomorphic to both the Q ij, Q1 i j and Q 2 i j. So, this is isomorphic to on the one hand this guy and on the other hand this guy. And let us see how to prove this lemma. This is fairly straightforward exercise.

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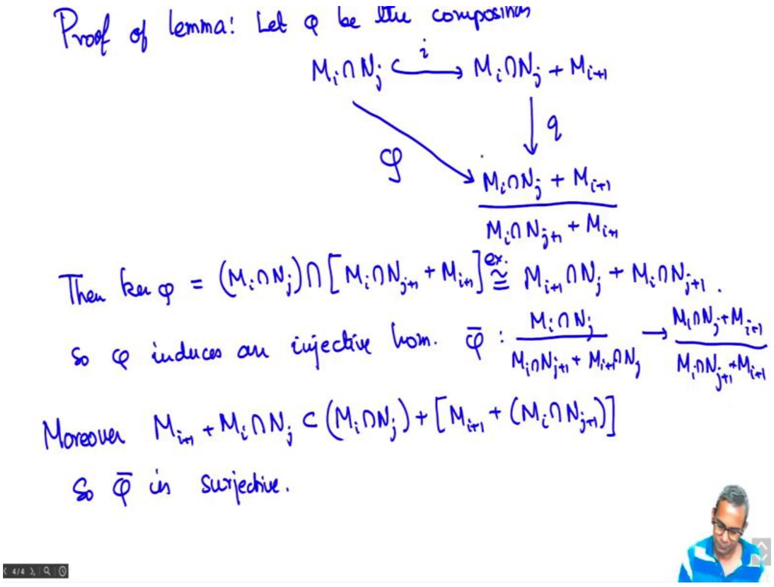

So, proof of lemma, what you do is, let us try to define an isomorphism for the modules on the left and the other one will be quite similar just interchanging the roles of M n i N j, role of M with N and the role of i with j. So, what you do is you have M i intersected N j. So, let us define, so we have what, we have M i intersect N j and this is a sub-module of M i intersect N j plus M i plus 1.

But that has surjective map onto the quotient module M i intersect N j plus M i plus 1 mod M i intersect N j plus 1 plus M i plus 1. So, this is the quotient map, this is the inclusion map and let phi be the composition of these two. So, we will call this phi and this phi will give rise to the isomorphism that we desire, we just need to check now all these things.

So, then what is the kernel of phi? The kernel of phi is all those things in M i intersect N j which also happened to be in this denominator here. And this you can show without too much difficulty that this is isomorphic to and I leave it as an exercise so you, that this is isomorphic to M i plus 1 intersect N j plus M i intersect N j plus 1.

And so, phi induces an injective homomorphism, phi bar from M i intersect N j mod M i intersect N j plus 1 plus M i plus 1 intersect N j on to M i intersect N j plus M i plus 1 modulo, you know this thing here. So, that is injective and now we only need to check that phi bar is surjective.

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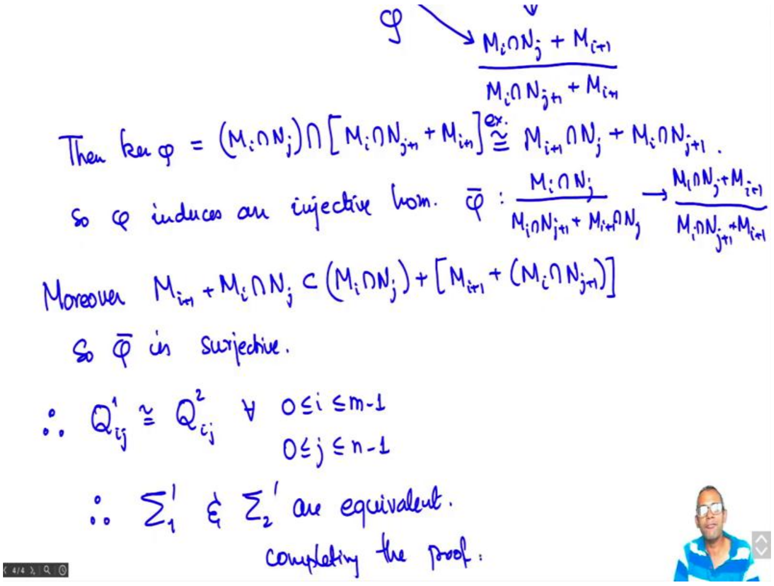

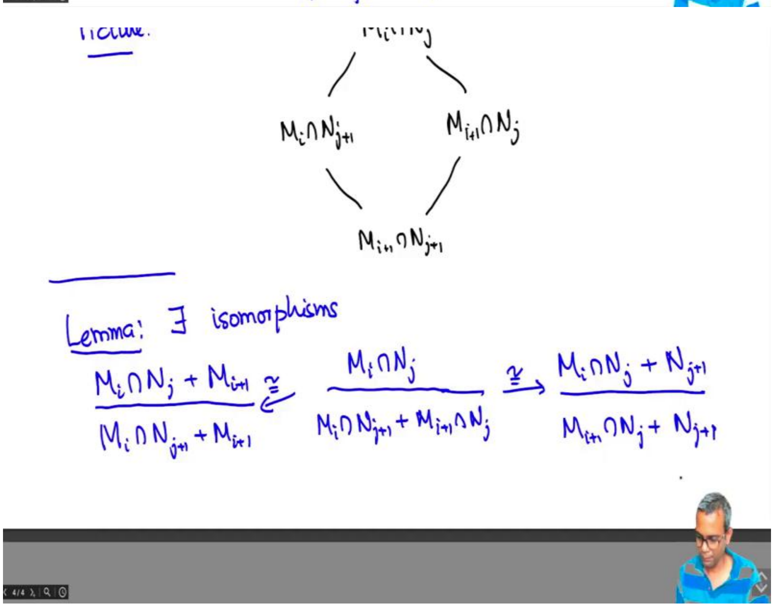

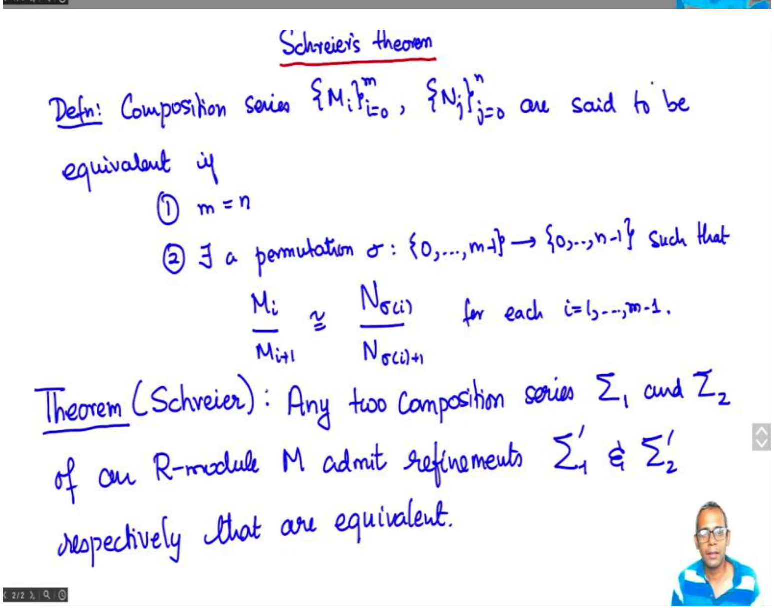

The image of phi bar is, so phi bar is surjective because, so phi goes on to this, the image of phi bar is the image of phi. So, we need to check that this is contained the image of this, but the point is that M i plus 1 plus M i intersect N j is contained in M i intersect N j plus M i plus 1 plus M i intersect N j plus 1. And so, because of this phi bar will be surjective. Because you are going to get everything over here modulo this in the image of phi. In fact, what this shows is that phi is surjective and therefore, phi bar is surjective.

So, what we have shown is that this thing, these two modules on the left are isomorphic and as I said before, by interchanging the role of M with the role of N and the role of i with the role of j, you can prove that these two modules on the right here are also isomorphic. And so, we have Q 1 i j is isomorphic to Q 2 i j for all i less than or equal to M minus 1. And then the remaining quotients are just 0 because the last module in the composition series in each row is the first module on the right.

And so, what we have is you can just remove those by merging those terms. Therefore, sigma 1 prime and sigma 2 prime are equivalent. Completing the proof of Schreier's theorem. And let me emphasize again, that in Schreier's theorem, there is no hypothesis when we prove the Jordan-Holder theorem, we will need a hypothesis about the ring R and about the modules M and N. Here this is a completely general theorem.

So, you can even look at rings, where the composition series are finite, but the rings can admit infinite composition series like Z. So, if you have any two finite composition series of Z they will admit, we are always taking composition series to be finite. That is how we define it, they will admit a common refinement. And not only does it say that, such a common refinement exists. The proof that we have given up Schreier's theorem actually shows you how to construct this common refinement.