Algebra – II
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Lecture 72
Composition Series

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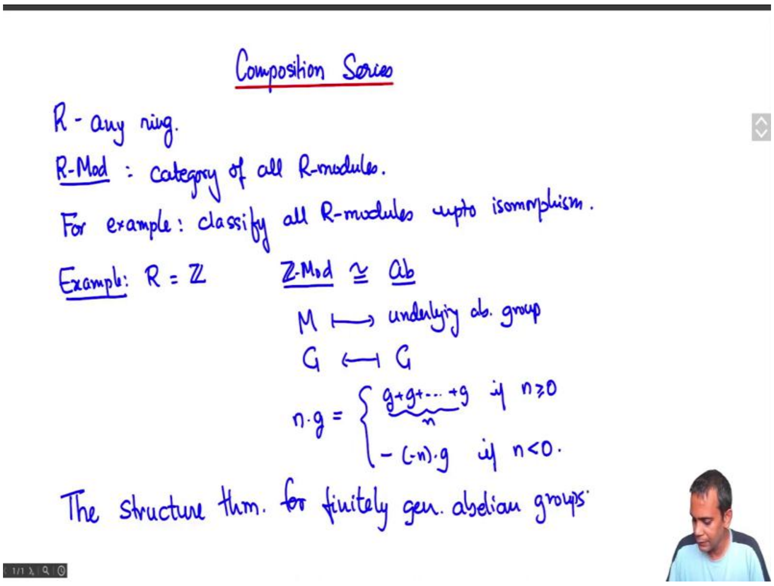

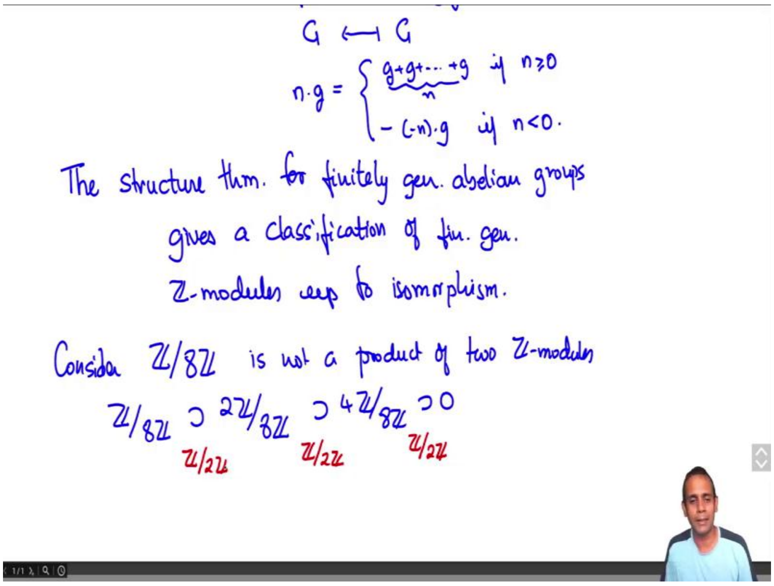

Today, I will tell you about composition series, our primary objective is to understand the category of modules over the ring. So, let R be any ring. And let us denote by R mod, the category of all R-modules. We want to understand this category. So, in some sense, we want to understand what are all the objects of this category. And one way to understand the category is to try to understand all its objects up to isomorphism.

So, for example, we may want to know how to classify all R-modules up to isomorphism. Now, this is usually a tall order, let us take even the simplest example. Let us take R to be the ring Z. And we know that R-modules are the same as abelian groups, the category R mod is isomorphic to the category, or ab of abelian groups. How does this work? Well, if you have an R-module M, you can just take it to the underlying abelian group.

And if you have an abelian group G, you can put the structure of, maybe I should say Z-module here, just to be clear, we are talking about Z, you can put on G the structure of a Z-module. So, how do you do it? You see, you need to say, given an integer n, and an element G of G, what is n dot g and you say that n dot g is g plus g plus g plus g taken n times, if n is greater than or equal to 0.

In particular, if n is equal to 0, then n dot g is 0, the identity element of the abelian group, and if n is negative, then you take it to be minus of minus n dot g, minus n would then be positive. And So, this would make sense by the earlier case, if n is less than 0. And so, an abelian group becomes a Z-module. So, the category of Z-modules is the same as the category of abelian groups.

And in algebra 1, we have seen a structure theorem for finitely generated abelian groups. So, what this does is, this gives a classification of finitely generated R-modules up to isomorphism. However, it turns out that if you remove this condition of finitely generated, then the classification is a not known in general, we do not know how to classify all Z-modules, or we do not know how to classify all abelian groups up to isomorphism.

So, in fact in algebra 1, we looked at the classification of finitely generated R modules, may be I should see here again Z, we did the classification of finitely generated R-modules for any PID R. And in some sense, that tells us a lot about the category of R-modules when r is a PID. So, we may take all R-modules, but usually that is quite unmanageable. We would take finitely generated R-modules.

Here, I do not want to get into specifics here I just want to motivate things. So, for example, consider the R-module, consider the Z-module, Z mod 8 Z. So, this cannot be rated as a product of two abelian groups or a product of two Z-modules, So, is not a product of two Z-modules. So, in some sense, it is a building block for the category of Z-modules.

But on the other hand, it is not the simplest object that you have this Z mod 8 Z contains, for example, 2 Z mod 8 Z, multiples of 2 inside it, and that in turn contains 4 Z mod 8 Z, but that is maximal submodule and there is nothing else. So, this kind of breaks up this Z-module into smaller and smaller pieces. And to get an idea of what is happening in between these two pieces, you can compute the quotient of Z mod 8 Z by 2 Z mod 8 Z and you will see that here this quotient is isomorphic to Z mod 2 Z.

Here this quotient is also isomorphic to Z mod 2 Z and here this quotient is also isomorphic to Z mod 2 Z. So, this is the kind of thing that we are going to be interested in the next couple of lectures. So, this Z mod 2 Z is an even more fundamental building block, this is what is called a simple Z-module. So, let us go to the definition of a simple module in general.

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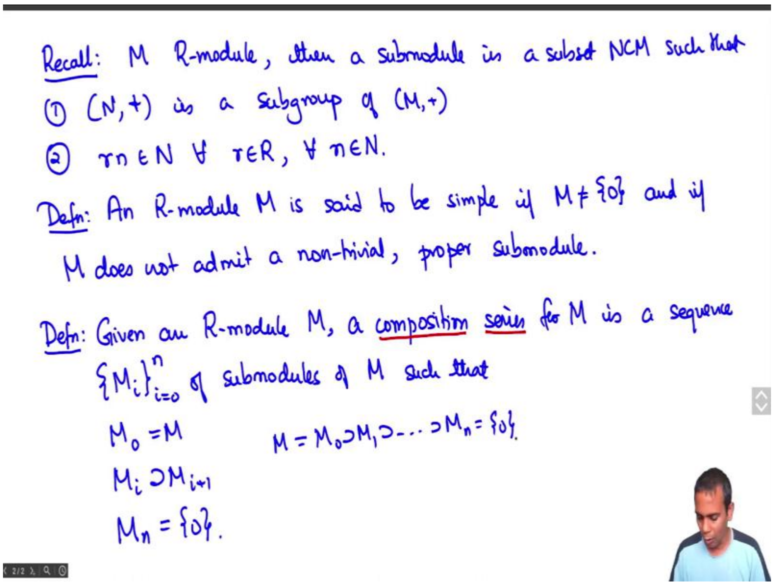

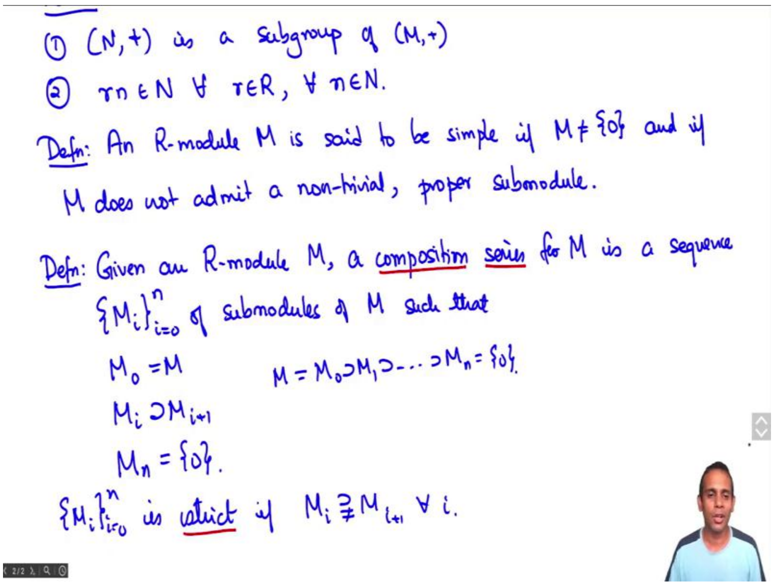

So, firstly, recall that if M is in R-module, then a submodule of M is a subset N of M such that firstly, we want that N under addition is a subgroup of M, the underlying abelian group of n is, so n has to be a module of course, and it has to be the underlying additive group has to be a subgroup of the underlying additive group of M and the second is that it should be a stable under all the operators coming from R. In other words, r n belongs to N for all r in R and for all n in N. In the category of Z modules, submodules are precisely subgroups.

Now, we can define a simple module, an R-module M is said to be simple, if it does not admit. Well firstly, if M is not just 0, that is it is a non-trivial module and if M does not admit a non-trivial, i.e., a non-zero proper submodule. So, for example Z mod 2 Z is a simple Z-module, whereas Z mod 8 Z is not a simple Z-module because it admits non-trivial proper submodules.

And now, we can define the main object of this lecture namely a composition series. So, given an R-module M, a composition series for M. So, this is what we are defining composition series is a sequence, a finite sequence we will take. So, M i i goes from 0 to n of submodules of M such that, well firstly, M 0 is equal to M, M i contains M i plus 1. And lastly, M n is the trivial module.

So, basically it is just a sequence, we have M 0 contains M 1 contains M n, there are n plus 1 things, the first one being M and the last one being 0. So, that is just the definition of a composition series. Sometimes you will be requiring a composition series to be strict. So, we will say that M i, i 0 to n is strict, if M i properly contains, M i plus 1 for all i. So, we often do not want to be distracted by composition series by terms just repeat.

And so, we will use strict in that case. And sometimes we want the flexibility of allowing composition series, where terms do repeat in which case, we will not insist on strict. Now, we can compare composition series, there is a partial order on the set of all composition series and this partial order is that of refinement.

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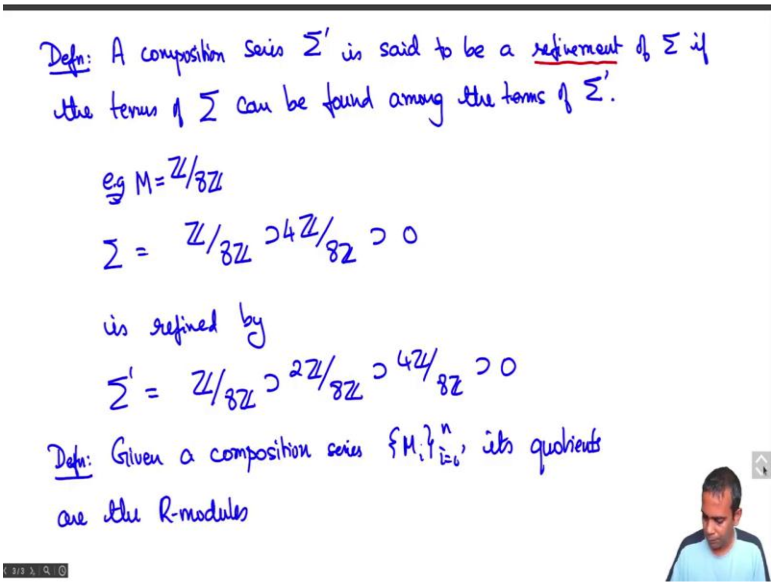

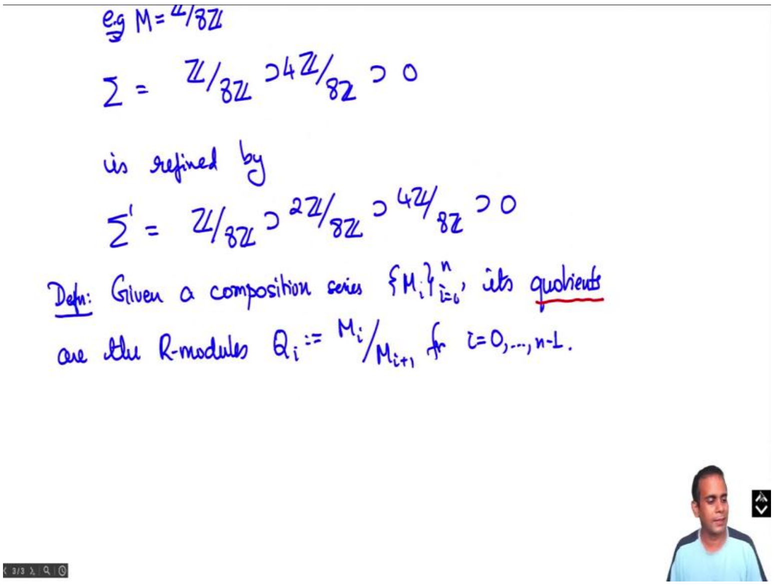

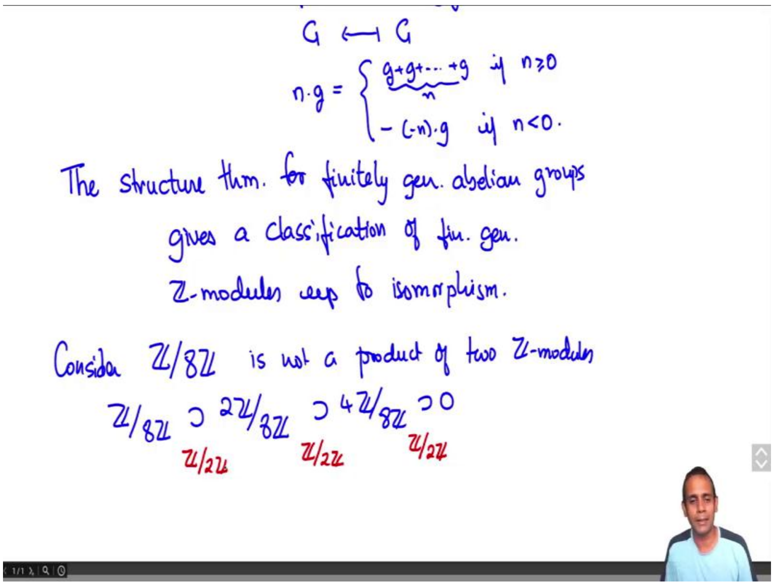

So, given two composition series sigma prime, a composition series sigma prime is said to be a refinement of sigma, of a composition series sigma if the terms of sigma can be found among the terms of sigma prime. So, for example, if we look at Z mod 8 Z, we have a composition series which is just Z mod 8 Z contains Z mod, let us say 4 Z mod 8 Z contains 0 is refined by.

So, refinement of this is Z mod 8 Z contains 2 Z mod 8 Z contains 4 Z mod 8 Z contains 0, because each of the terms of this first series sigma is to be found amongst the terms of the second series sigma prime. So, we will say that sigma prime is a refinement of sigma. And one last definition related to a composition series; given a composition series, let say M i, i goes from 0 to n, its quotients are just the R-modules.

Q i defined to be M i mod M i plus 1 for i equals 0 up to n minus 1. So, these are the quotients of a composition series. So, we have already seen some examples. Here, for example, I have written down the quotients of this composition series in red. So, let us just study all these concepts using an example.

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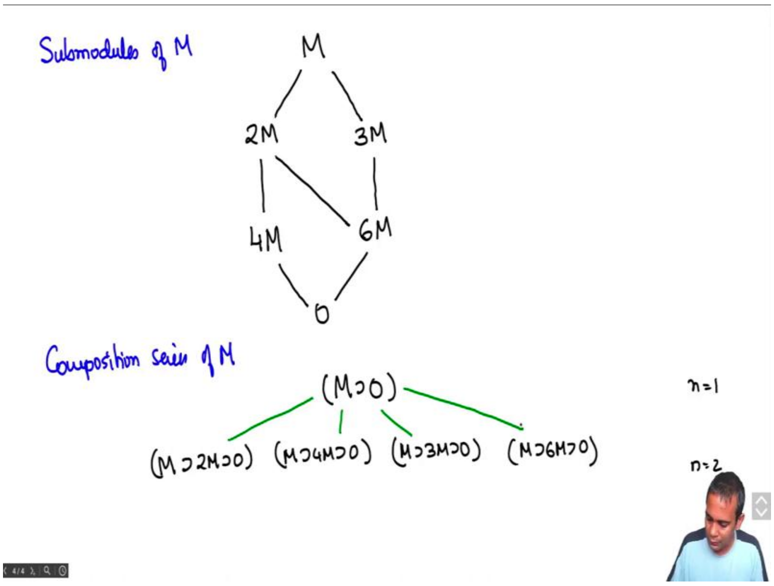

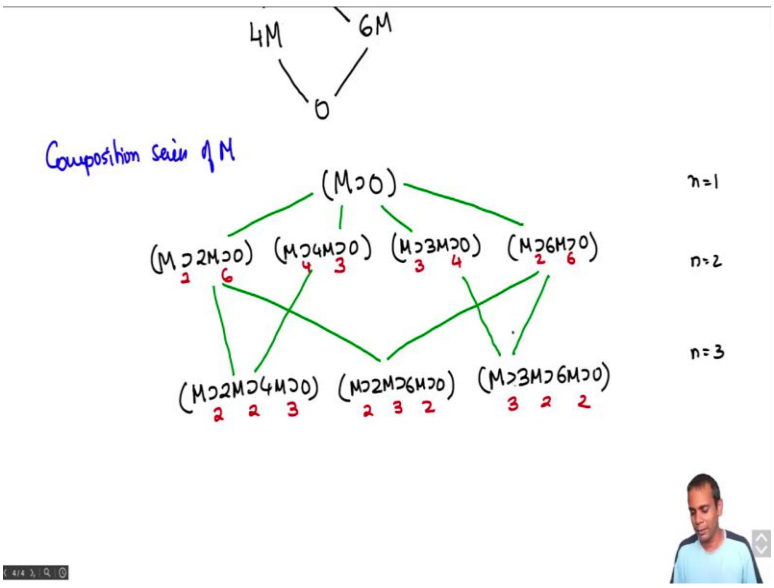

So, example, take R to be Z. So, we are just talking about abelian groups. And let us look at M equals Z mod 12 Z, this is an R-module. In order to write down the composition series, for M, we will first find it helpful to write down all the submodules of M. So, let us first list all the submodules of M. I listed somewhat visually, so that you can see all the different submodules. So, for example we have, so the largest submodule is Z mod 12 Z itself, let me just call it M.

And then this contains two submodules, the multiples of 2, which I will call 2 M, and the multiples of 3, which I will call 3 M, it also contains the multiples of 4, which I will call 4 M, but the multiples of 4 are contained inside the multiples of 2. And then there are multiples of 6, which are contained inside multiples of 3. However, multiples of 6 are also contained inside multiples of 2. So, this diagram, and then there is the trivial module.

So, these are all the submodules of M, there are totally 6 of them, including M and the trivial module. And from this, we can write down all the composition series, they just got to be sequences where you choose modules going down in chains in this diagram, but you do not have to take everything along a chain. So, you start with M, and you end with 0, and you can take a path going from M to 0, and pick some things along that path.

So, the smallest composition series that you can have, is just the composition series, which has two terms that is N is equal to 1, you have M contains 0. So, this is a composition series with n equals 1. So, let us just, now I am writing down the composition series of M. And this every composition series will be a refinement of this most trivial smallest composition series. And now this composition series will contain two term composition series, or rather maybe I should say three term composition series, but n equals 2.

So, this will contain, well one for each non-trivial submodule of M. So, you can take M contains, 2 M contains 0, you can take M contains 4 M contains 0, you can take M contains 3 M contains 0, and you can take M contains 6 M contains 0. So, these are the composition series with three terms, or n equals 2. And I will just draw a line to indicate that each of these refines the trivial composition series M contains 0.

And now, we have the next stage where n is equal to 3. So, this composition series M contains 2 M contains 0 can be refined by adding a term 4 M. So, here we have M contains 2 M contains 4 M contains 0. So, this is certainly refinement of this composition series, but it is also a refinement of this composition series. Now, how can we refine M contains 3 M contains 0?

We can refine it by adding M contains 3 M contains 6 M contains 0. And this is going to be a refinement of these two-composition series. But there is one more composition series where you take this diagram and go across from the left to right. So, M contains 2 M contains 6 M contains 0. And this is going to be a refinement of M contains 2 M contains 0, and M contains 6 M contains 0, but it is not going to be a refinement of these two guys in the middle and that is it.

So, this diagram shows you all the composition series of Z mod 12 Z, and it also shows you the refinement relations if you go from up to down along the green line, then you get a refined composition series. And so, you see there are three composition series, which are maximally refined, which cannot be refined any further. Now just for fun, let us also write down the sub quotients here.

The quotients of this, so here, we get 2 Z mod 2 Z, I will write down the size of the sub quotients, they are all cyclic groups. So, there is no ambiguity there. And 2 M is like, it is a order 6, so this here is 6. Out here, you have 4, and you have 3 as a sub quotient, out here you have, sorry, I think this one is, this one is 3, and 4. And out here, you have 2 and 6. So, these are the orders of the quotients of these series.

And let us continue that down here. So, here you get 2, 2, 3. Here, you get 2, 3, 2 and here you get 3, 2, 2. So, you see, there is a nice pattern also in terms of the sub quotients, one thing that you observe here is in all these composition series at the bottom of this diagram, the sub quotients are the same, but they appear in different permutations. So, we will be interested in composition series of this last kind, the most refined ones, the ones that cannot be refined any further and those are called Jordan-Holder series.

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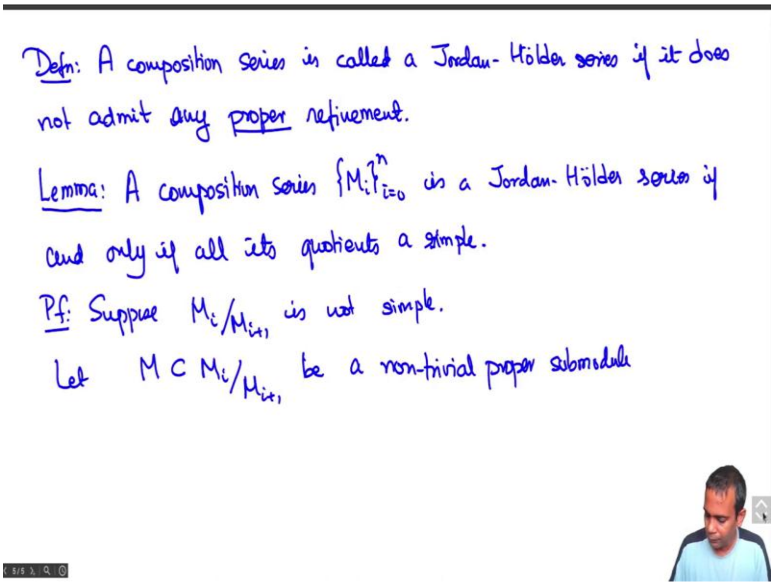

So, definition, a composition series is called the Jordan-Holder series, if it does not admit any refinement. But well, a series technically is a refinement of itself. So, I should say any proper refine, if it does not admit any proper refinement, whereby proper I mean, refinement apart from itself. So, you must add at least one term. And one way to, another way to characterize Jordan-Holder series is in terms of the quotients.

So, that is simple lemma that a composition series M i, i goes from 0 to n is a Jordan-Holder series if and only if all its quotients are simple. Let us see how to prove this. The proof is fairly simple. So, suppose, so we have two ways to prove. So, suppose one of the quotients is not simple, so suppose, M i mod M i plus 1 is not simple. So, what that means is that we have a proper submodule, a non-trivial proper submodule, be a non-trivial proper submodule.

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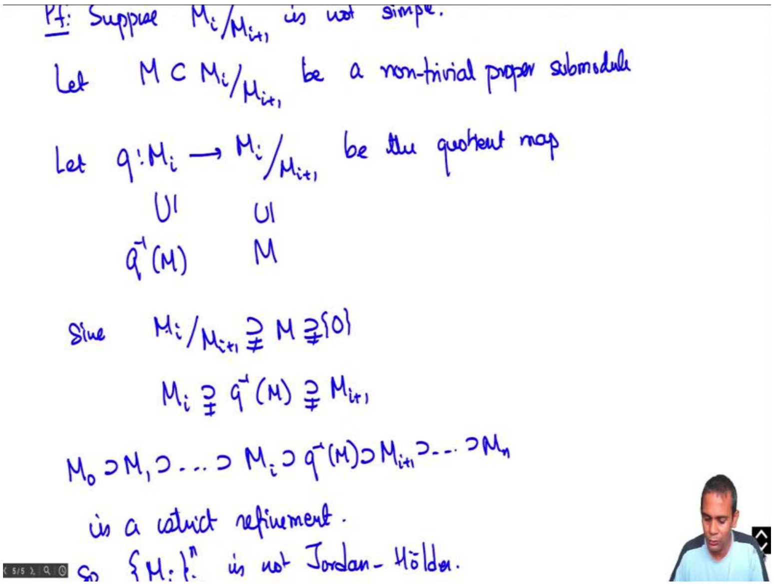

Then, what you can do is let q be the quotient map from M i to M i mod M i plus 1, and in here you have M which is a submodule and you can look at q inverse M which is contained in this. Now, since, M I strictly contains, well I guess since, M i mod M i plus 1 strictly contains M and that is strictly non-zero, e have that M i strictly contains q inverse M and that strictly contains M i plus 1.

So, now you can write down a strict refinement, what is, which is M 0 contains M 1 contains and then you take M i and then you put and insert this new term q inverse M in between and then M i plus 1 up to M n is a strict refinement. Which means that the original series was not Jordan-Holder. So, that proves one way, it proves that if the series is Jordan-Holder, then every quotient is simple.

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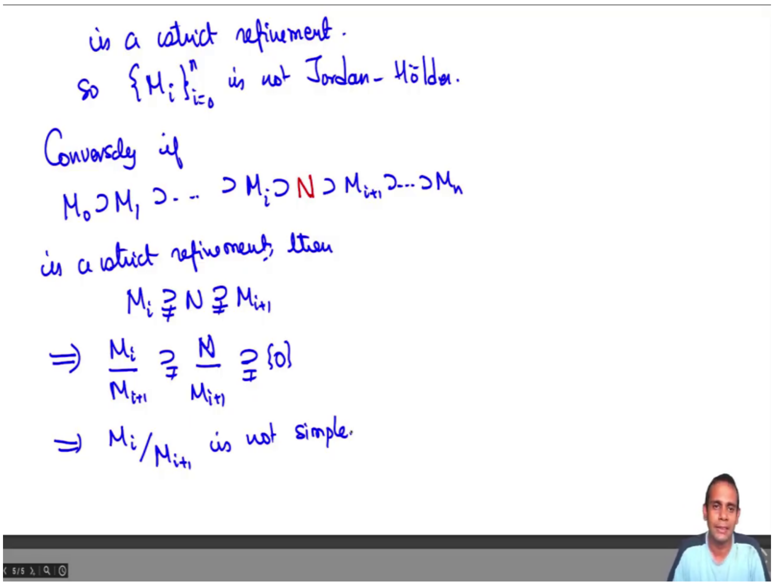

Now, conversely, we will show that if there is a refinement. So, if we have a refinement M 0 contains M 1 contains M i, and then we have a new guy which we insert, let us call it just N contains and then we go back to our old series, is a strict refinement. Then what we are seeing is that, so I think when I say strict refinement, I need to be a little careful, the terms, there is a term in this series, which is not one of the terms in the other series.

So, what that means is that this N is neither equal to M i nor equal to M i plus 1. So, what we have is M i strictly contains N which strictly contains M i plus 1, which implies that if you look at M i mod M i plus 1 this strictly contains N mod M i plus 1 and that strictly contains 0 which means that M i mod M i plus 1 is not simple.

So, this shows that if all the quotients are simple, then you cannot further refine a composition series. So, now in the next lecture I will show you that Jordan-Holder series is essentially unique in a certain sense, and we will prove that theorem, it is called the Jordan-Holder theorem.