**Algebra – II**
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**Lecture 71**
**F-algebras**

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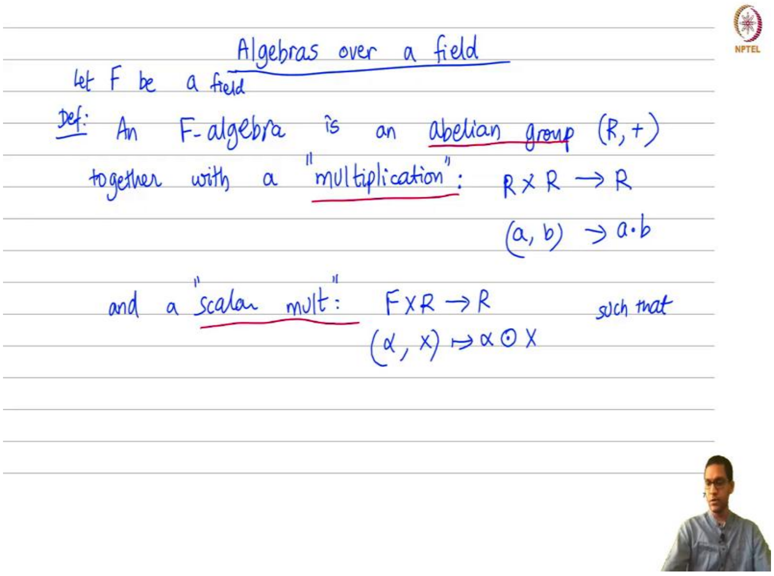


Today, we will briefly talk about the definition of another algebraic construct. So, this is called an algebra over a field F. So one can more generally define algebras over a commutative ring, but for now let us just restrict ourselves to algebras over fields. So, what is an algebra? Well, it is the following, let F be a field, let us fix a field F. So, an F-algebra is an abelian group R plus together with a multiplication.

So, what is a multiplication? A multiplication is a map, it is binary operation if you wish, let just call it dot. So, multiplication is a map, R cross R to R, which takes a pair a, b to a dot b, it is a binary operation. And so, with a multiplication map, and a scalar multiplication map. So, what is the scalar multiplication map? This is a map from the field cross R to R, given a scalar alpha from the field and in element X from R, it maps to some element of R.

So, it is just alpha, if we put a circle dot X. So, I am given two maps at the moment, I will tell you what the axioms are. But at the moment, I am just calling these maps by these names, multiplication map and a scalar multiplication map, such that the following axioms hold, So, recall the three things, it is an abelian group, it is got a map like this, R cross R to R, it is got a map from F cross R to R, such that the following things are true.

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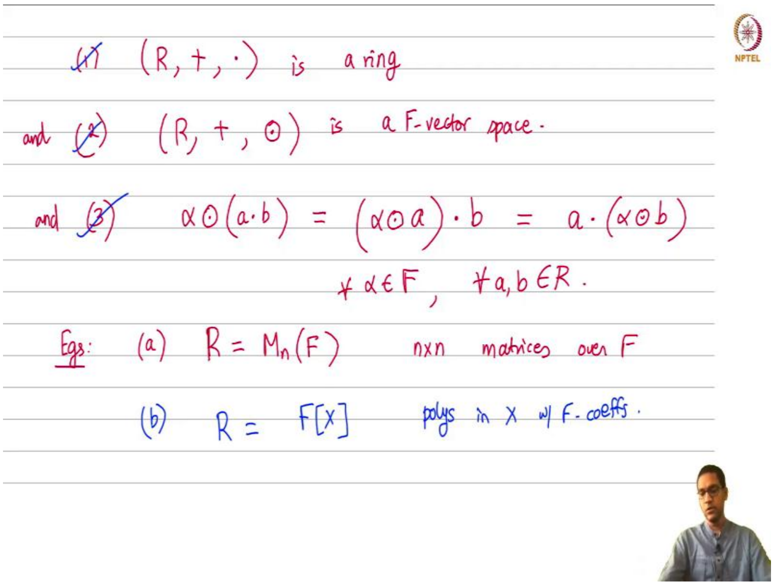


Number1, if you look at R with plus and multiplication, then this is a ring. Property 2, and you look at R as an abelian group together with the scalar multiplication, this becomes a vector space over F, F-vector space. So, in some sense, therefore, it is ring and the vector space, but the two operations sort of are compatible.

And thirdly, the compatibility axiom, which says that, if your scalar multiply a scalar alpha with a product of two elements of the ring, then this is the same as first scalar multiply a, then multiplied by b in the ring or multiply a with the scaled version of b, this should be true for all scalars from the field, for all elements, a b from R. The third axiom is really the compatibility axiom.

Now, what are examples? Because why is one studying this concept? Well, there are a large number of very interesting examples. So, here are two of them. So, if F is the fixed field there, I have fixed F for now, I can look at the set R of n cross n matrices over the field, this is all n cross n matrices over F. Now, observe what is the multiplication and scalar multiplication.

So, recall this is of course, it is a vector space over the field F. Now, you just scalar multiply by multiplying every element of the matrix by the given scalar. This is also a ring it is got a ring structure, which is usual matrix multiplication. And the key point is that, these two structures, the scalar multiplication and the multiplication satisfy the compatibility axiom.

This axiom is definitely true, because it just says if you take a product of two matrices and then multiply by a scalar on the outside, it is the same as you multiply the first matrix by the scalar and then multiply the two matrices together or multiply the second scale the second matrix and multiplied by the first one. So, this is of course, just a well-known property for scaling and multiplication for matrices.

Number2, is well you can take polynomial rings. So, here this ring structure is not commutative. But here, the F-algebra F X, which is all polynomials with coefficients in F, this is just the polynomials in X, one variable with F coefficients. So, this is of course, a ring, again, you have multiplication, but it is also got a vector space structure, you can multiply a polynomial by a given field element. And here these, it is easy to check again, the compatibility is true. So, these are all examples of F algebras. And so, these three axioms of an F algebra, 1, 2, and 3 can also be written in a slightly different form.

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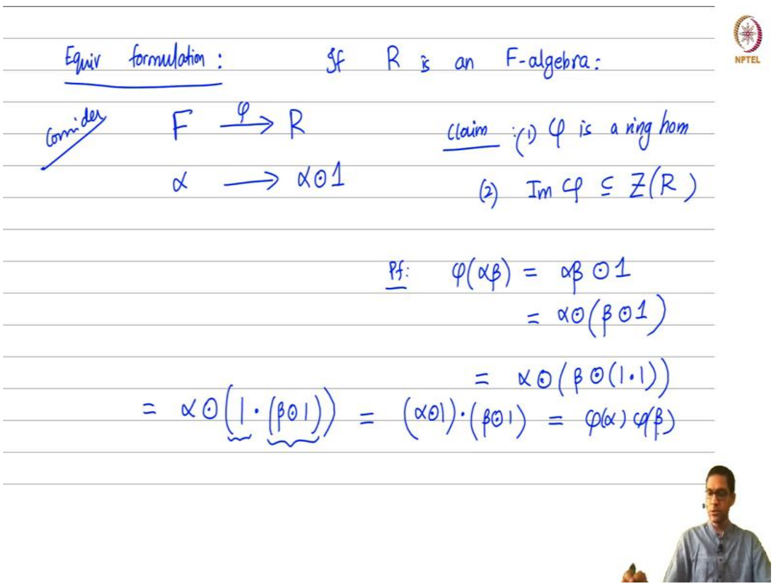


And so, this is sort of an equivalent formulation, if you wish. So, observe that if I am given an F algebra, so, observe, so before I tell you what the formulation is, let us make some observations. If R is an F algebra, what does it mean? Well, it says the following, I know how to scalar multiply elements of R by elements from F. So, consider the following map, take a scalar alpha and map it to I mean, the ring has a special element the multiplicative identity, you scalar multiply it by alpha k, you can scale 1 by alpha and that gives you some ring element.

So, consider the map. Now, what sort of map is this, call it phi, consider phi defined like this then claim is that phi is a ring homomorphism, that is property 1. Claim, property 2 is that the image of this phi is contained inside the center of this ring R. So, I claim here are two special properties that phi is a ring homomorphism and its image is contained in the center of the ring R.

So, why is this true? Well, both are easy to check. Let us check the first property for example. So, if I have, so what is phi of alpha beta? By definition, it is alpha beta scalar multiplying 1. But how do you scalar multiply alpha beta with 1? Well, one way of doing this is to say, well, let us say it is alpha, beta scalar multiplying 1 into 1, because the multiplicative identity is just one times itself.

And now, so alpha beta is a scalar. So, what I can say is that, because of the vector space axiom, when I multiply, so maybe we will do that in the next step just for clarity. So, when you have alpha beta multiplying 1, this is just alpha scalar multiplying beta scalar multiplying 1. This is just the vector space axiom, if you wish. But what is this? This is alpha scalar multiplying. So, beta scalar multiplying, I will write this as1 dot 1. Now, let us go here, this is alpha multiplying. Now, let us use one of the axioms that says this is beta times1 dot 1 is just the same as1 multiplying beta dot 1. So, this was the compatibility axiom.

And now finally, think of this as some ring element, this as some ring element and alpha scalar multiplying that is the same as alpha multiplying the first element, and that multiplied by the second element. And so this is exactly phi of alpha phi of beta, which shows that it is in the one of the properties of the ring homomorphism. So, again, you have to check phi of alpha plus beta similarly.

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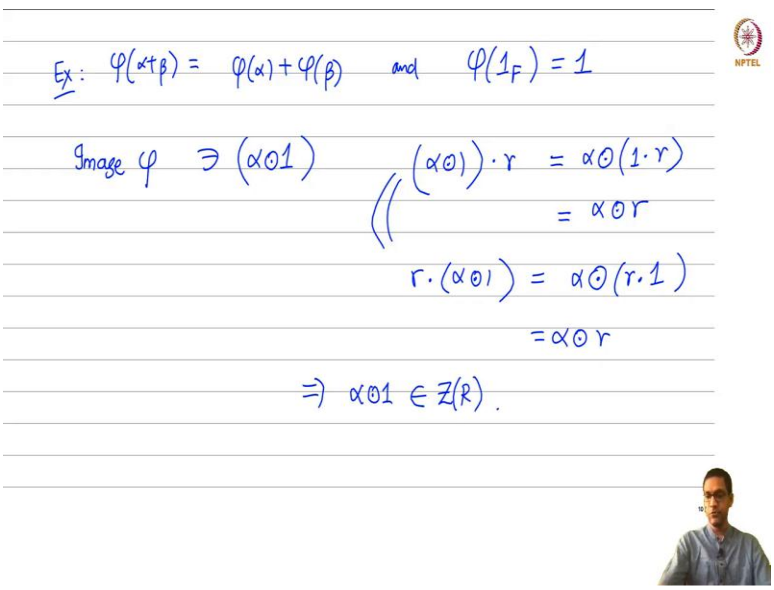

So, let me leave that as an exercise to similarly check phi of alpha plus beta equals, exercise. And the unit element of the field maps to the I mean, this is the easy part, the identity of the ring. So, now, what this means it is the ring homomorphism. Now, why is the image inside the center, that is a second property. So, let us check the image of phi. So, what is a typical element of the image? It looks like alpha multiplying 1.

Let us take a typical element alpha multiplying scalar multiplying 1. This is a typical element of the image, I claim that this element must be in the center, in other words, if I take this and multiply it by some ring element R, question is what do we get? Well, what do we get, again by the compatibility axiom alpha times1 dot R is just alpha scalar multiplying 1 into R, which is just alpha times R.

Whereas, if I multiplied in the other order r times alpha 1. Again, the compatibility axiom says this is just, but that is just alpha times R. Therefore, these two things are equal to each other. In other words, alpha scalar multiplying 1 commutes with the element R. So, this means. So, what this means is that if you have an algebra if you have an F-algebra then that comes from that you can define a map phi from the field F to the center of R.

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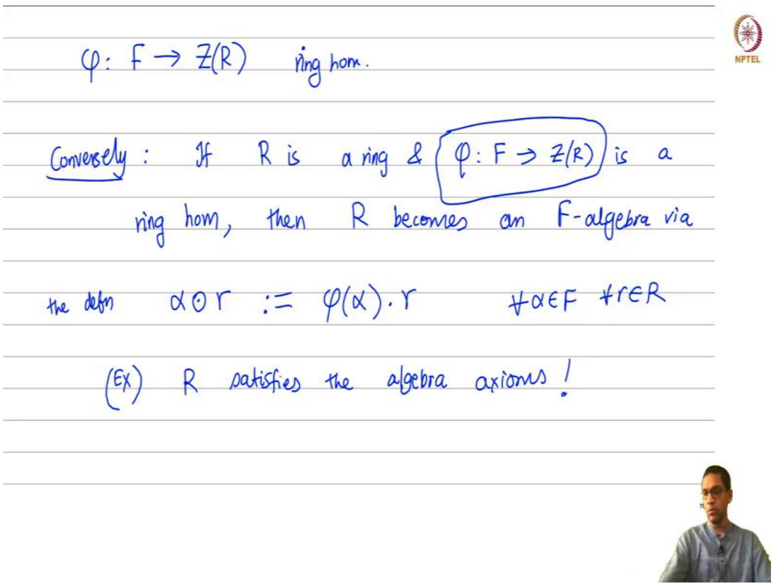


So, this phi the ring homomorphism, from F to, you can construct this. But in fact, conversely, so actually this is giving such a map is equivalent to giving an F-algebra structure. Conversely, if I have an F-algebra or given such a map, conversely, if R is a ring homomorphism, then R is an F algebra, R becomes an F-algebra via the following definition of scalar multiplication. What is the definition?

Given any element alpha, how do I and an element R of the ring, how do I compute alpha scalar multiplying R? I define it to be, I first apply this homomorphism phi to alpha that gives me some ring element, I just multiply that by r. So, via this definition, alpha scalar multiplying r is phi alpha times r. Now, you can check that this satisfies all the axioms. Exercise, check that R satisfies the axioms of an algebra. So, F-algebras are really the same data as homomorphisms from F to the center. So, why is one wanting to do this?

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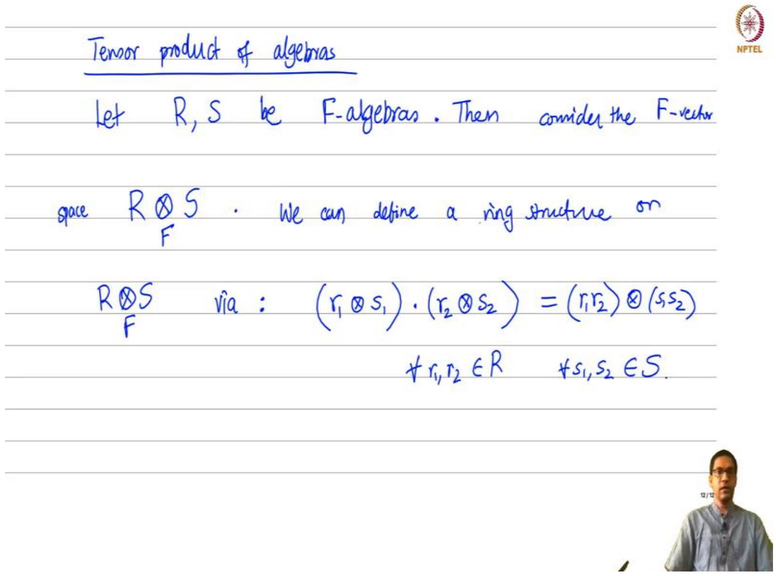


Well, number1, algebras are ubiquitous they occur a lot. And there are many interesting things one can do with algebras. For us, of course, the immediate motivation is the tensor product of algebras. And this is an important construction also. So, suppose that if I give you two F-algebras, let R and S be F algebras. Again, F is the field that I fixed before.

Then what can I do with it? Well, I can construct, so recall, F-algebra means, I have got a ring structure, or well I have an abelian group structure, there is a multiplication and there is a, or rather there is a ring structure and a vector space structure which are compatible with each other. Now, at the moment, forget the ring portion, let us only think of them as vector spaces over the field F.

So, I have R and S are both vector spaces over the field F. So, I can, therefore modules over the field, I can talk about that tensor product. So, consider the F-vector space, F-vector space R tensor F over the field F. So, it is a vector space definitely. But I claim this vector space actually comes also with a ring structure. So, how shall we define our ring structure on this? So, recall R and S have multiplications.

So, let us define, we can define the ring structure. Well, how can we do this? Well, again, let me just tell you what the final answer is in terms of the decomposable tensors. What we want is this, we want, so let us take an element a simple tensor of the form r1 tensor s1. And multiplying by r2 tensor s2. So, well, I wanted to be the following, I would like it to be, I can multiply r1 and r2 in the ring r and s1 and s2 in the ring s.

So, look at r1 r2 tensor s1 s2. And this is what I want to be true for all r1 r2 in R, s1 s2 in S. But of course, as always, I cannot use this as my definition per se, because then I will have to check well defined and so on. So, what one does instead is again, we try to do it more abstractly using the universal properties So, that you will be guaranteed that there exists a well-defined map like this.

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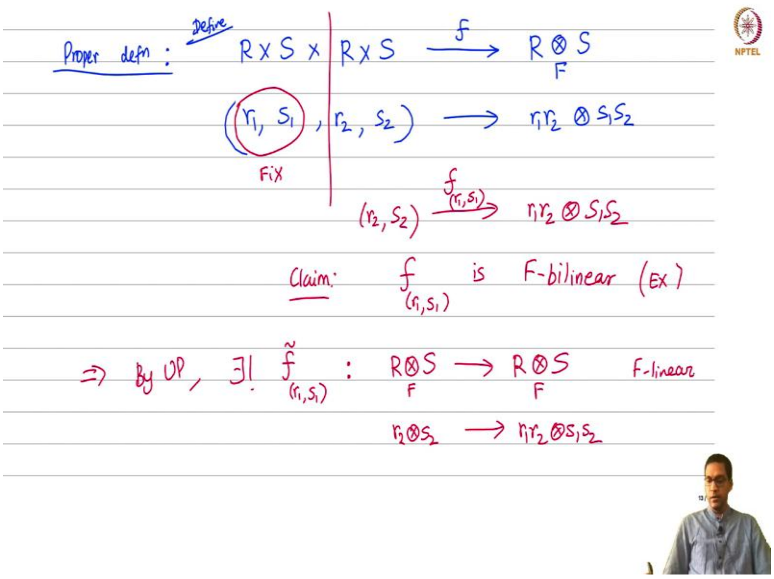


So, let us try and give a proper definition. So, again, I will leave most of the verifications as exercises. So, let us sort of go back as far as we can. So, instead of I mean, what is this a map from this is, the multiplication map is really recorded some map from R tensor S cross R tensor S to R tensor S. So, when I say ring structure, I mean a multiplication map. So, let me define instead the following.

From R cross S cross R cross S, I will define a map. So, let us define a map F as follows. So, what is F2? Well, it takes tuple r1 s1 (())(17:10) it to r1 r2 tensor s1 s2. So, I can always construct this map. Now the key point here is we need to sort of get the correct bi-linearity and so on. So, to do that, let us say the following. Let us fix r1 and s1. So, you fix this, these two components, think of it only as a map from R cross S to R tensor S.

So, this fixed map here, so let me call that map as F. So, this overall map is called F, but this map which I get when I fix an r1 and s1. So, this map it sends r2 s2 to, let me give this a name, I will call it F sub r1 s1. So, I am fixing r1 and s1, and I get a map like this. Now, what properties does this map have? So, the first claim is that this map is in fact, F-bilinear. So, F sub r1 s1 is an F-bilinear map.

So, again, check exercise. Just plugin the change r1, the change r2 s2 and see what happens to the answer. So, it is F-bilinear therefore, by the universal property, there exists a unique map, let us call it f tilde, f tilde but sub r1 s1 still depends on r1 and s1. This is now a map from R tensor S. And what does this map do? Well, it takes r2 tensor s2, maps to r1 r2 tensor s1 s2. So, there exists at least it is a well-defined map like this and what sort of map is this? This is an F-linear map. Now, however, this depends on an r1 and an s1 which we have fixed.

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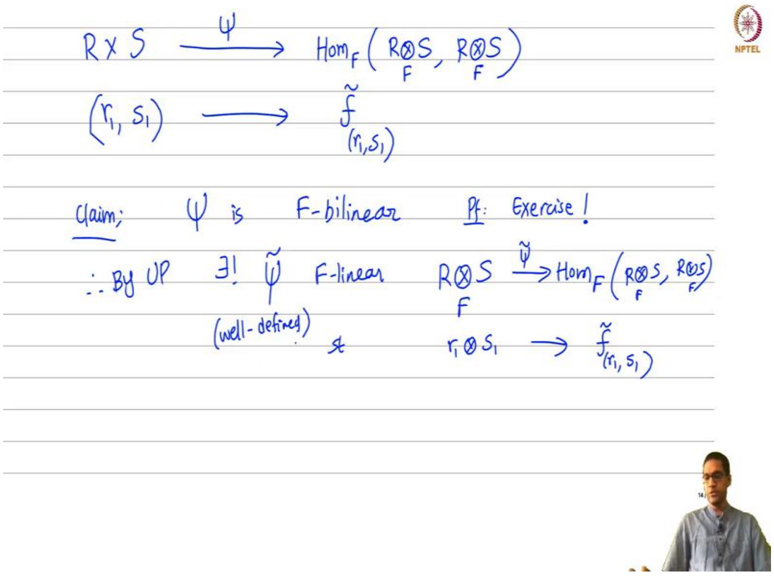

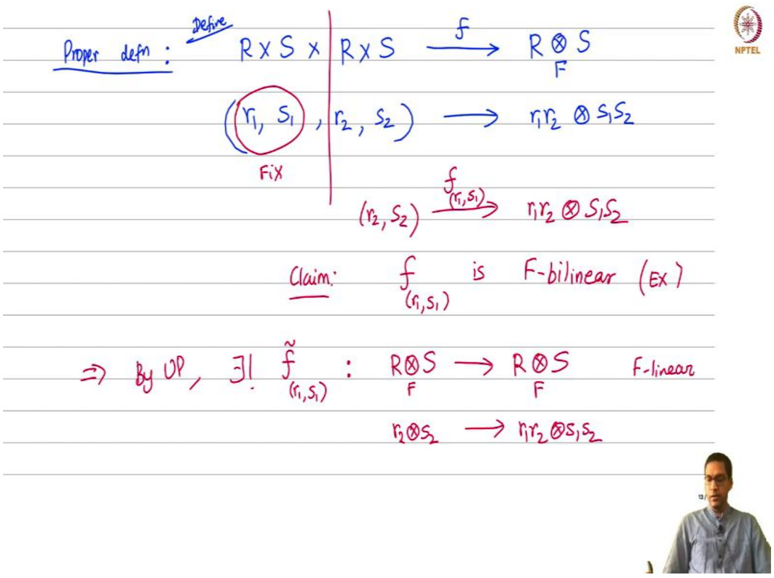


So, we now try and remove that dependence as follows. So, look at the following, from R cross S, so for each pair r1 s1, I have now defined a map called f tilde r1 s1. Now, where does f tilde belong? Well, f tilde was a linear map. So, it is in the space of all F-linear homomorphisms from R tensor S to itself. If you go back and see that is exactly what it does. It is a map from R tensor S to R tensor S, and it is an F-linear map.

So, f tilde r1 s1 belongs here to the space of homomorphisms. So, now you get a new map, I know we should call it something else psi maybe. And now claim again, psi is also bilinear, psi is F-bilinear. Which means when I change r1 and s1 I should see how f tilde r1 s1 changes. So, again prove exercise, check bi-linearity it. Just a question of plugging in the definitions.

Now, again therefore, by the universal property there exists a unique psi tilde, this is known F-linear map from where to where, from R tensor S to the space of F-linear homomorphisms such that this diagram commutes. So, I should say, such that what does it do? It takes r1 tensor s1 to f tilde of r1 s1. And that is more or less what we want finally.

So, now once you have a map like this, you can use this this map to define your, this is exactly the map you want, this is the map psi tilde. So, now we define a multiplication using this map. So, psi tilde is a well-defined that is what we finally wanted, that is the key here. So, I should say there is a unique key point is, this a well-defined map which does this.

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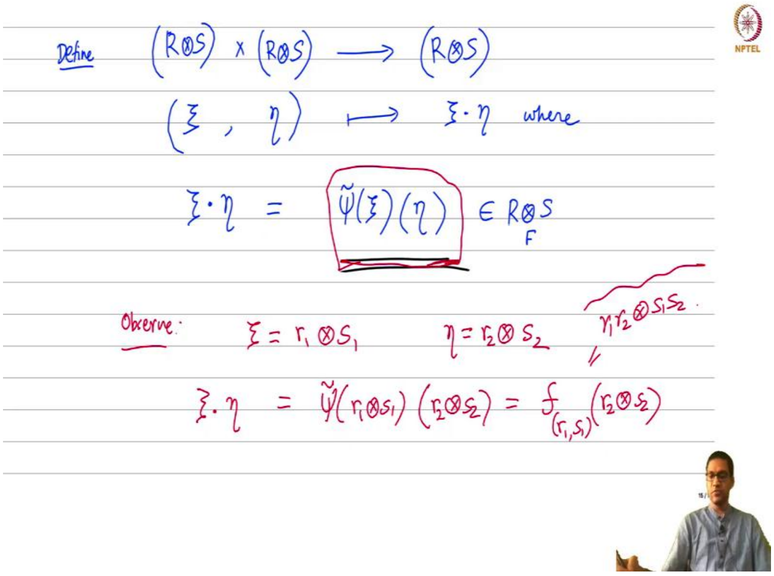


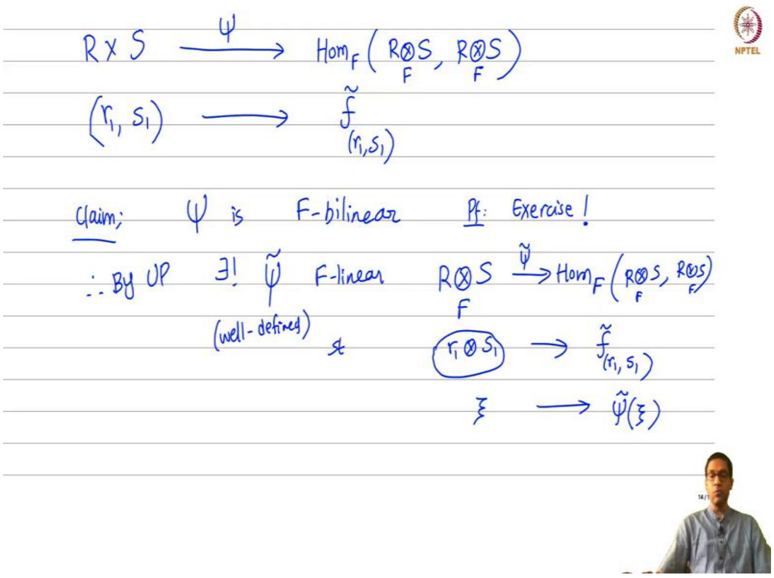


So, now we use this map. Therefore, now we define a map from R tensor S. I am going to find the multiplication map now. So, how am I going to find the multiplication map? So, I am just going to say take for each r1 tensor s1, I mean, maybe one should just say, I mean, I want to it more generally not just going to it for the r1 and s1. So, take xi and eta, xi is an element from R tensor S, eta is another element from R tensor S.

How am I going to define their product? So, I will define their product to be the following. It is just, so maybe I should say, this goes to xi times eta, where I define xi times eta, as follows. What am I supposed to do? So, I take the psi tilde map. So, I know what psi tilde does to, such elements, but in general, it will take any xi to something, it will take it to some element psi tilde of xi.

Ad that new elements psi tilde of xi is now some endomorphism of R tensor S. So, here is what I will do, I first apply psi tilde to xi, the answer is now some homomorphism from R tensor S to itself. So, I can evaluate it on an element of R tensor S, I evaluate it on eta. And what I will get will be some other element of R tensor S. So, this is my definition, finally. So, once I get this map psi tilde, I can define multiplication as follows. It is psi tilde of xi evaluated on eta.

But now, observe that this definition has the following desirable property, that if xi and eta are both decomposable tensors, if this looks like this, then, what is this new definition I have made? Well, it is just going to be, I evaluate psi tilde on this, evaluate the answer on this. But then we just saw psi tilde is nothing but f sub r1 s1 evaluated on r2 tensor s2, that by definition is exactly r1 r2 tensor s1 s2.

So, that what it says is, this is the correct definition. I mean, this is the formal way to define it, So, that it is well defined and so on. But on the decomposable tensors, it behaves the way we want it to behave. That is why we define the map like that in the first place. So, this is sort of, this is the amount of work it took just to define the multiplication. But having done this, we still have to finally show it as an F-algebra.

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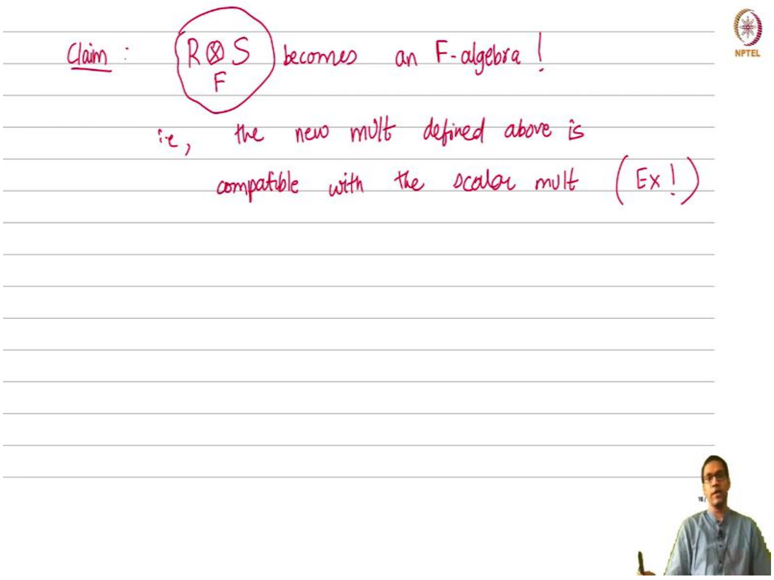


So, my final claim is that if I take R tensor S over F becomes in fact an F algebra with respect to this new multiplication, I have defined becomes an F algebra. In other words, the new multiplication that we just defined, defined above is compatible. So, that third compatibility axiom is true with the scalar multiplication. So, I am going to leave this also as an exercise.

So, what this finally does is, is to give you a powerful way of constructing new algebras from old ones. So, I mean one can also, if I take instead of fields algebras over fields, I can just take algebras over rings, commutative rings. So, if I replace F by say the ring of integers, then Z algebra is the same thing as a ring. And this notion here is like the notion of tensor products of rings. So, given two rings, what you do is more or less follow the same sort of thing to construct a new ring, whose operation is this interesting component wise product and then taking tensor.

So, I can define the tensor product of two rings, for instance, like this. So, this is a rather powerful construction. And, for example, in the category of rings, this is exactly the co-product. So, this is a, you can think of this as an application of this whole tensor products thing that we have been doing, it gives you a way of constructing, products or co-products of rings, for example.