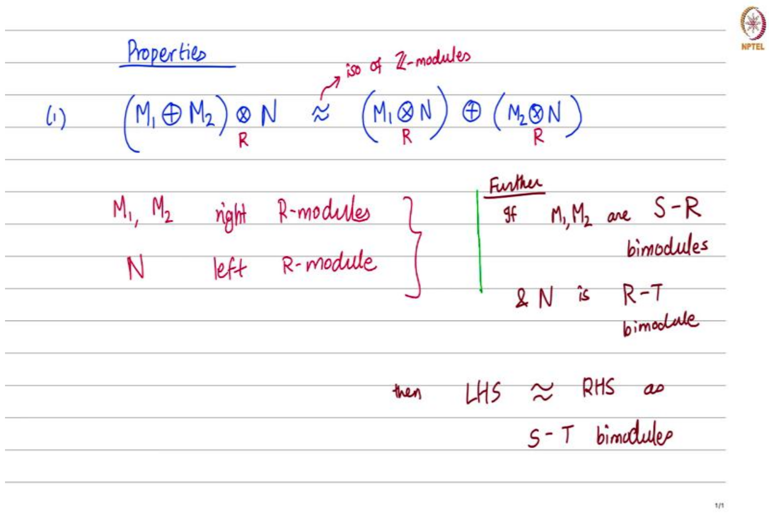
Algebra – II
Professor S Viswanath
Mathematics
The Institute of Mathematical Sciences
Lecture 70
Some Properties of the tensor product

(Refer Slide Time: 00:14)



Let us talk about some key properties of the tensor product. So, here is the first property. So, some properties. So, the first one concerns what happens when you take tensor products of direct sums of modules. So, let me tell you what the statement is, suppose I take a direct sum of two modules, M1 direct sum M2, and then try to tensor the resultant with N, then the claim is that this is the same as isomorphic to M1 tensor N, this is one module, direct sum M2 tensor N.

So, of course, now when I say this, we need to be a bit careful with, what is it a module over and so on. So, let us say this is a tensor product over the ring R. So, what sort of objects should M1 and M2 be? So, I should assume M1 M2 are right, R-modules, and N is a left R-module. So, this is the very least I require, only then can I actually construct this tensor product even.

And when I say, isomorphism, what do we mean by isomorphism. So, observe both left and right hand sides are well, they are Z modules in general, right abelian groups, and this is an isomorphism of Z modules. In other words, it is a group homomorphism, between these abelian groups, it is an isomorphism of Z modules. So, this is the first property.

And of course, this is sort of only the bare minimum. But, recall, we sort of said, typically, if you had not just right or left modules, but rather some bimodule structures. So, more generally, you could also have this statement phrased as follows. So, suppose, if M1, M2 are not just right R-modules, but let us say there are some ring S, for with it is a left module as well as, so it is an S-R bimodule.

And N is R-T bimodule, so here R, S, and T are rings, then still the same, the both sides, the left-hand right-hand sides here, they also become bimodule. In fact, they become S-T bimodules, then the isomorphism that we talk about then left-hand side, so let me write like this, left-hand side is isomorphic to the right-hand side, as not just Z modules, but as S-T bimodules.

This is further, so this is an additional thing. If I also had this additional structure, then this isomorphism also respects that additional structure. So, let me just indicate briefly how the proof goes and leave it as an exercise for you to carry out the details. So, what does one have to do? One has to construct maps from the left-hand side to the right-hand side and in the opposite direction, such that their composition is the identity. So, let us try and construct maps.

(Refer Slide Time: 03:50)

want $(M_1 \oplus M_2) \otimes_R N \longrightarrow \cdots$

Instead: consider $(M_1 \oplus M_2) \times N \xrightarrow{f} (M_1 \otimes_R N) \oplus (M_2 \otimes_R N)$

$$((m_1, m_2), n) \longrightarrow (m_1 \otimes n, m_2 \otimes n)$$

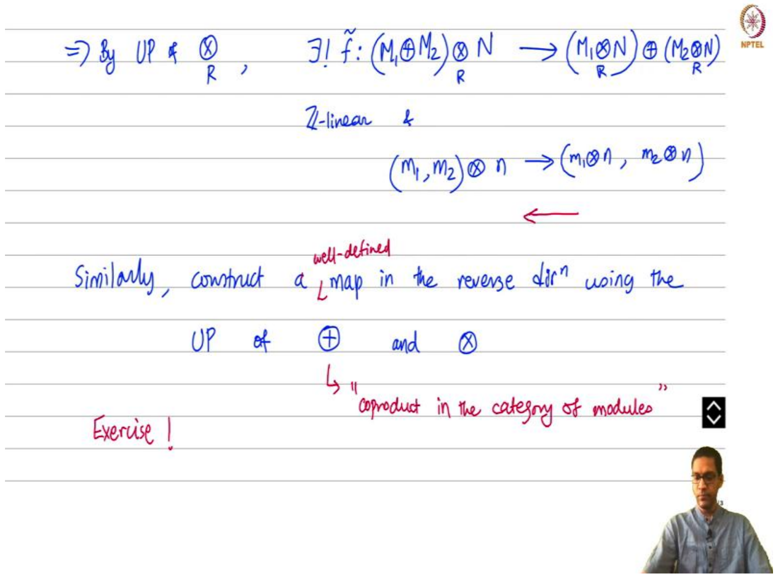
Check: f is $\mathbb{Z}$-bilinear & R-balanced.

So, how does one do it? So, let us see. So, let us try to construct a map. So, you somehow want a map from a M1 direct sum M2 tensor N to some other module, to the right-hand side. Now, whenever you want to construct a map from a tensor product to something, what that really means is you should try and construct a R balanced Z bilinear map from M1 direct sum M2 cross N to the right-hand side.

So, since I know that I am trying to get a map from the tensor product, I will sort of back up one more step, and instead try to construct a map. So, instead, let us do the following. Let us consider M1 direct sum M2 cross N to the right-hand side, let us try to construct a map f, right-hand side recall is M1 tensor N, direct sum M2 tensor N. And more or less, the definition itself is, so this is M1 comma M2, that is an element here.

And from n, if I pick an element so it is like I am taking a triple of elements M1, M2, and n. And from that I need to somehow construct an element in the right-hand side. So, there is one obvious choice I just do M1 tensor n. So, by the way, all these tensors are over R comma M2 tensor n. So, recall, the direct sum is as a set, it was just the cross product. So, let me define the map in this way and then we just need to check the following facts which I leave for you to check as an exercise. f is Z bilinear and R-balanced. So, check these two properties. Now, once you do that, what it implies is by the universal property.

(Refer Slide Time: 06:00)



want $(M_1 \oplus M_2) \otimes_R N \longrightarrow \cdots$

Instead: consider $(M_1 \oplus M_2) \times N \xrightarrow{f} (M_1 \otimes_R N) \oplus (M_2 \otimes_R N)$

$$((m_1, m_2), n) \longrightarrow (m_1 \otimes n, m_2 \otimes n)$$

Check: f is $\mathbb{Z}$-bilinear & R-balanced.

2/2

So, once you do this, then this implies by the universal property of the tensor products, there exists a unique map f tilde from M1 direct sum M2, tensor N to the right-hand side. This is what sort of map is, it is a Z-linear map and which does the following with sends M1 comma M2, tensor n, to whatever the map f sends it to, which is M1 tensor n.

So, what you have managed to do is to use the universal property to construct a map in one direction. And then, by a similar token, similarly, construct a map in the reverse direction. And I mean, you sort of know what it means to do, we want the composition to be the identity. So, just show that there will always exist a map in the reverse direction by sort of, again using the universal property of direct sums and tensor products.

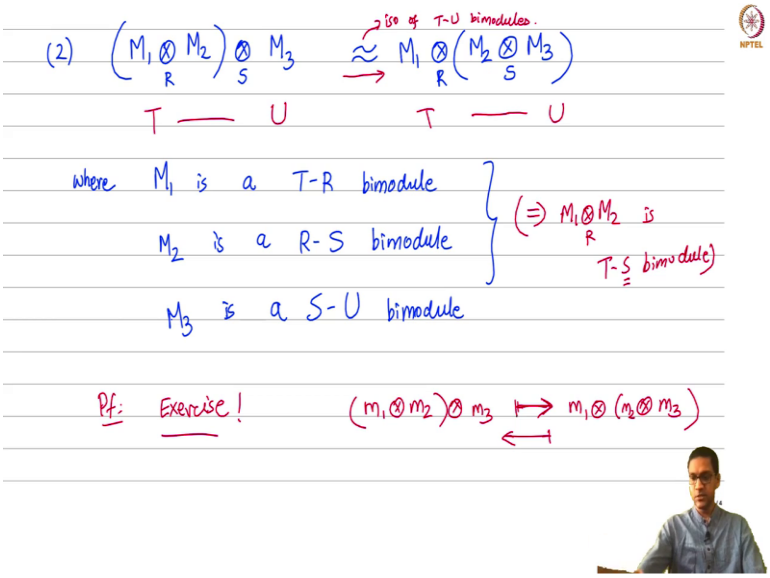
So, let me just say, using the universal property of the direct sum, and the tensor product. So, recall by the universal property of the direct sum, so recall the direct sum you learned in categories and functors. This is just the co-product in the category of modules over, so in this case, they are both right R-modules if you wish, so this is or Z modules. So, it depends on what you are talking about.

But in general, depending on what ring is the underlined ring, it is a co-product in the category of modules over any given ring, So, a co-product, of course, has this universal property, that if you have maps from the two modules, some other module, then that map factors uniquely throw the co-product. So, use that and use the universal property of the tensor product.

So, I am just going to leave this as an exercise, it is fairly routine. We sort of know what the map needs to be, it is just that you need to show it is a very defined is really where the key.

So, in all these things with tensor products, it is well defined (())(09:01), that one needs to work a little bit for. So, this is the statement here, that if you take the direct sum tensor N, it is M1 tensor N direct sum M2 tensor, as Z-modules or more generally as S-T bimodule.

(Refer Slide Time: 09:19)



Let me state property number 2. Again, I am not going to prove this, but leave this as an exercise instead. So, suppose I have a tensor product of three modules. So, suppose I do M1 tensor M2 over R, and then I take the tensor product of the answer over a third module. So, now here in general, this is over some other ring in general S. So, I want to construct I mean, I want to consider such tensor products, threefold tensor products.

Where, and now let us just do everything in the bimodule situation most generally. So, what do I want M1 to be? Where M1 is a, so it is definitely a right R-module and, on the left, I need to have some module, so maybe I will call it T1, maybe we will just call it T, so it a T-R bimodule. So, I am just putting in the minimal requirement, it is a T-R bimodule.

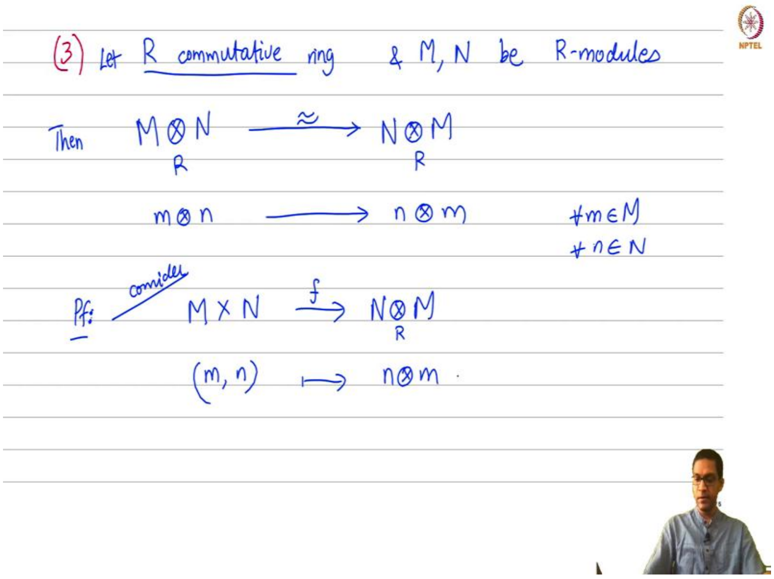
What is M2? M2 is a R-S bimodule. So, suppose I do this then observe this implies in particular that M1 tensor M2 therefore, M1 tensor M2 is a T-S bimodule. Again, so since it is a T-S bimodule, T-S on the right I can further tensor product it over S with some other module M3, provided M3 is a S-U bimodule, let us say. So, I suppose I am given M1, M2, M3 like this, then this sort of iterated tensor product makes sense.

And my claim is that, this is isomorphic to, so sort of like an associativity property, I first tensor product M2 and M3 together over S. Now, the resulting answer is what bimodule is this? Now, M2 was in R, this is an R-U bimodule now the tensor product, and now I tensor it over R with M1, so the resulting answer will become a T-U bimodule. So, both sides observe are T-U bimodule.

So, the claim is that this is an isomorphism of T-U bimodules. Again, I am not going to prove this, proof exercise again, it is the map itself is more or less, you will have to construct the map similarly by trying to construct Z bilinear, R balanced maps and so on. But the idea itself is that you should map this element here to the corresponding element M1 tensor M2 M3 tensor. So, the claim is that, this is what the map finally does the isomorphisms.

But to, again, you cannot use this as the definition because a well-defined (())(12:51) will become a problem. So, you go back to the definition, use the R-balanced maps and bi-linearity and so on and the universal property of tensor products to actually construct these maps in a well-defined fashion. So, this isomorphism that you are looking for does this. So, proof again exercise, so you have seen this many times.

(Refer Slide Time: 13:18)



Properties

(1) $(M_1 \oplus M_2) \otimes_R N \overset{\text{iso of } \mathbb{Z}\text{-modules}}{\approx} (M_1 \otimes_R N) \oplus (M_2 \otimes_R N)$

M_1, M_2 right R-modules
N left R-module

Further if M_1, M_2 are S-R bimodules & N is R-T bimodule

then LHS $\approx$ RHS as S-T bimodules

Now, property number 3. Again, which is, but this is special for commutative rings. So, let me just say R is commutative. So, this is sort of a commutativity property if you wish. So, you can think of the first thing as being some kind of distributivity property for the tensor product, tensor products distribute over direct sums. The second is an associativity property of tensor products. And the third is a commutativity property of tensor products, but that only holds if the underlying ring R is commutative.

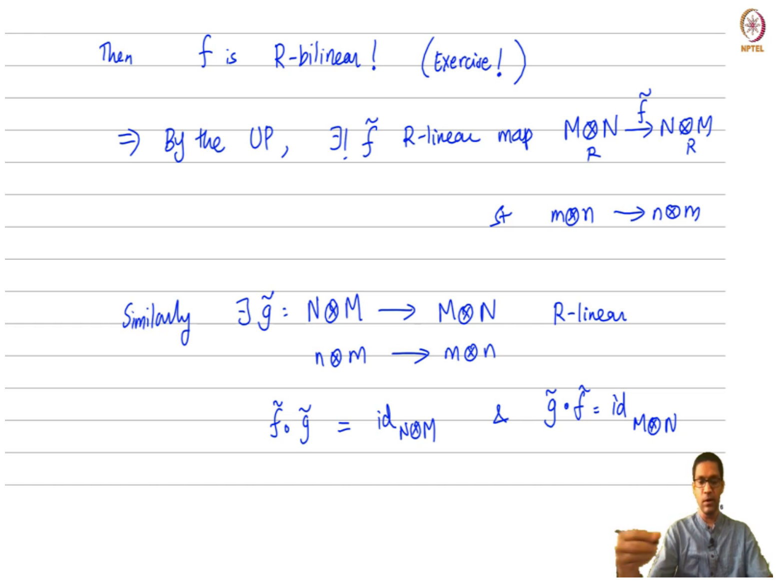
So, let the R be commutative ring and M, N be R-modules. Then, now, why do I want this? I want to be able to construct M tensor N, as well as N tensor M. So, observe in order to talk about M tensor N in general, I only need M2 be a right R-module, and N to be a left R-module, but then on the right-hand side, I have them in the opposite order. So, then for the right-hand side to make sense, I will need M2 be a left R-module as well, and N should be right R-module.

So, at the very least for both left and right-hand sides to make sense I will need both M and N to be R bimodules at least. But that is not enough in general just having sort of left and right, our action is not enough, compatible actions. I need those two actions to be the same action, only then will this isomorphism work here. So, then the claim is that these two things are isomorphic.

In fact, we are saying something more, the map, the isomorphism sends the element m tensor n to the element n tensor m, for all m in M and n in N. So, recall such elements generate the two tensor products. So, claim is that, these elements M tensor N are just being mapped to the corresponding elements M tensor N. So, again, this is a special property for commutative rings. And, again quickly let us just prove this.

In this case, recall since R is commutative the tensor product, I talked about this earlier, can be thought of as having another universal property, which is that if you take R bilinear maps. So now, in order to prove this, let us do the following, let us consider from M cross N to N tensor M, I will define the following map m comma n going to n tensor m. We consider this map let us call it f.

(Refer Slide Time: 16:13)



Now, observe the key property of f is that f is R bilinear, then f is R bilinear. So, recall this was an additional property, there is not just R balanced, it is a little bit more for commutative rings. So, why is it R bilinear? Well, because of the way the actions are, so again, proof exercise check that, this is R bilinear, and here is where you will use the commutativity of R.

Now, because f is R bilinear, which implies by the universal property, and this is the extra universal property, which holds for the commutative case, there exists a unique f tilde, what sort of map is this? R-linear map from M tensor N to, such that it maps m tensor n to. And that is all we need. Because observe f tilde itself is its own inverse, if you do it twice, it is clearly the identity map.

So, I mean, maybe I should not call the other map f tilde, so maybe there is a map, which does a similar thing in the opposite direction. So, maybe we should say, switch the roles of M and N. Similarly, that is really our f tilde inverse, so there exists g tilde from N tensor M to, so this is again a R-linear map. And then observe that f tilde and g tilde are clearly inverses of each other. So, this is the identity on N tensor M. So, what this says is that, that is sort of a commutativity property as well. So, these are three important properties in some sense for the tensor product functor that we have constructed.