Algebra – II
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Problem Session – Tensor Products of Vector spaces

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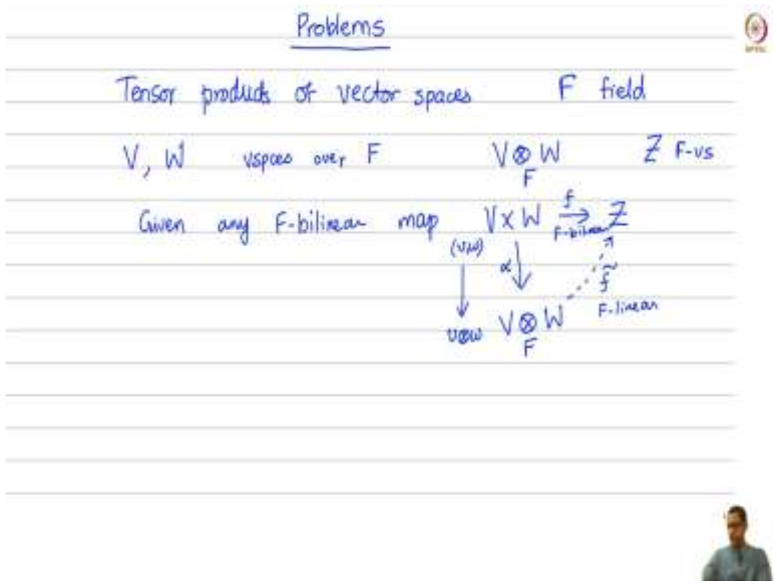

Let us, do some problems. So, specifically on tensor products of vector spaces, so recall vector spaces are nothing but modules over a field f, so let us fix a field f. So, field is necessarily a commutative ring, so recall we have an alternate universal property for the tensor product in this case. So, what was the tensor product?

If I give you two vector spaces V and W, then both vector spaces over F the spaces vector spaces over the field F, the tensor product V tensor W over F has the following universal property you can define it as the object which has this universal property which is that given any bilinear map any F bilinear map from V cross W to any other to any F at a space Z, given any F bilinear, map F where Z is again, so take Z again to be F vector space, then this map gives rise to a unique F linear map.

So, this is now an F linear map, the original guy was F bilinear and this map here now is just a standard map with just takes every pair V comma W and maps it to the corresponding element of the tensor product, what we called alpha before. So, that is just a recall of what the tensor product was.

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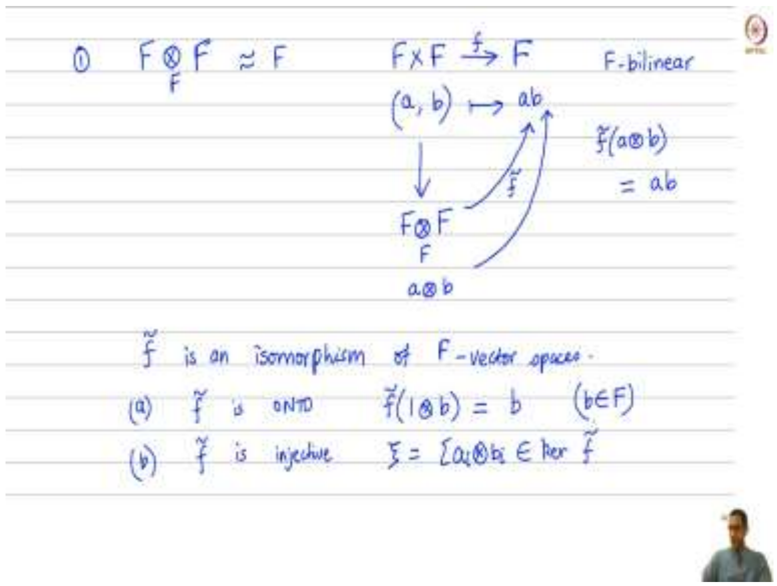

So, observe, so as an example, so let us do maybe problem 1. So, let us take just the vector space F itself, one dimensional vector space over itself. Now, I claim that if I look at F tensor F over F and this just (())(02:19) F. So, let us I mean this is an instance of a general thing we have proved before, but let us any we prove this from scratch. So, I have to try and construct a map from F tensor F to F.

So, the correct way to start is to first construct a bilinear map from F cross F to F and this is easy, if you just take a pair of elements a and b in F, I can just define their product, a b in F and observe the product map is of course bilinear. So, observe that this is clearly F bilinear and it is easy to see because of the distributive T property in the field.

Now, what that means is that this gives rise to a unique map from F tensor F over F to F, so there is a unique map F tilde and this map well what is this map do? It takes the element a tensor b of F and gives you the element a b, this is because this is it makes this diagram commute, so observe on the generators on the elements a tensor b, this just gives you the product a b.

Now, the claim is that this map f tilde is actually an isomorphism, f tilde is an isomorphism of f vector spaces, it is easy to see that it is onto so that is the easy part f tilde is onto because every element of the field can just be obtained as the image of this special element, just look at 1 tensor b that is just an element b. So, I can just take, you know just let b run over all possible elements of F and the image of the element 1 tensor b is exactly b.

Now, why is this 1 to 1 f tilde a claim is also a 1 to 1 map or injective map and to prove this, let us consider a typical element, so let us take an element xi equals summation ai tensor bi in F. So, this can also be written as so I assume that xi is in the kernel of this map and my goal is to prove that xi is 0.

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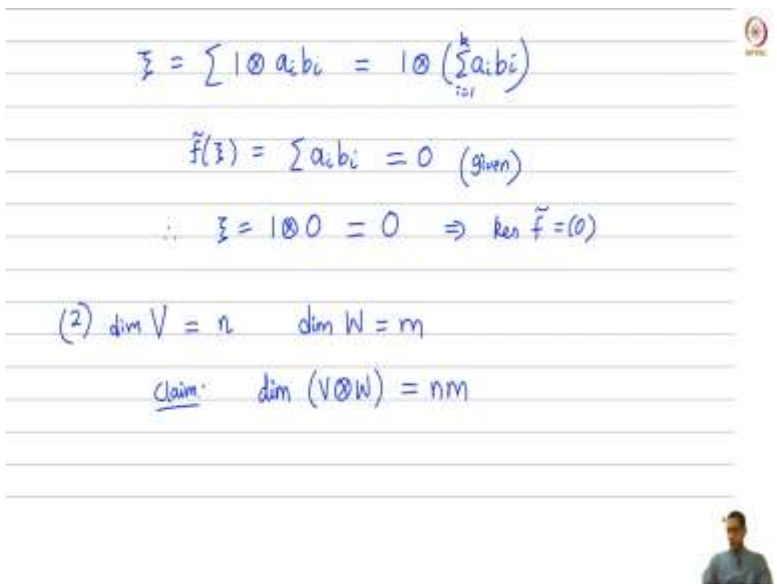

So, xi can also be re-written as well I can push the ai over to the other side, if you wish this the R balanced condition, so right you can push it from one side to the other, so I am thinking of the ai in the first component as 1 times ai. And now I use the fact that this symbol is bilinear I can write it like this. And what is f tilde of this? Therefore, f tilde of psi is of course by definition summation ai bi.

And that is given to be 0, so (())(05:28) assume that psi is in the curve. So, what does that mean? It means summation ai bi is 0, but then psi is of the form 1 tensor 0, therefore what it means is psi is just 1 tensor 0 and this we have proved before that anything tends to 0 that element is always the 0 element of the module by just writing 0 as 0 plus 0 and so on.

So, I would not repeat that argument, observe that psi is essentially becomes 1 tensor 0 and therefore that is just the 0 limit. So, that shows that the kernel of this map is just trivial, in other words the map is 1 to 1. So, that is that completes the first example F tensor F is actually just F. Now, more generally, what if you took vector spaces of other dimensions? Let us say finite-dimensional vector spaces.

So, let us say F is a finite dimensional, so let us even say it has some dimension of, V is a vector space of dimension n, W is a vector space of dimension m, so all vector spaces over the field F, then claim is that V tensor? W is in fact a finite dimensional vector space whose dimension is just the product of the two dimensions. So, you had seen this back in algebra 1 as well, but let us redo this from our point of view.

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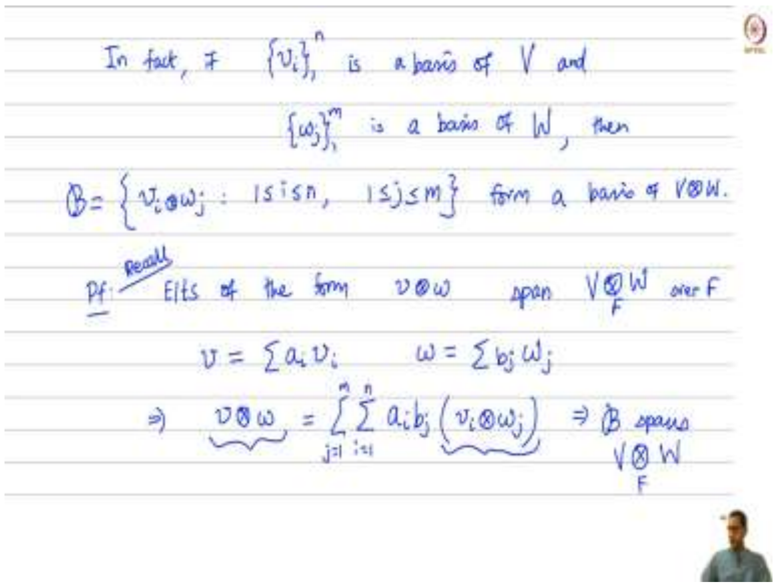

Now, in fact more is true in fact if you just take a basis if vi is a basis of V and wj so basis of W then we claim that the elements vi tensor wj these nm elements, 1 less than equal to i less than equal to n, they form a basis of V tensor W. And that will of course show that the dimension is n m. So, let us prove this stronger fact that these guys form a basis, so observers firstly that they certainly span, why do they span?

So, observe that elements of the form just a single v tensor single w like what we call the simple or decomposable tensors elements of this form certainly span V tensor W. So, recall this fact, so this just came from more less the construction of the tensor product. So, these fellows definitely span over F if you wish we span this space over the field F.

Now, any such element can be re-written as follows if I write v as the linear combination of the vi and w as a linear combination of the wj's, then by the bilinearity of this tensor symbol V tensor W will just become summation ai bj, vi tensor wj, this is now this is now i going from 1 to n, j going from 1to m.

So, what this means is that these elements vi tensor wj they certainly span the space, because V tensor W's are all expressible as a linear combination of the vi tensor wj's and V tensor W's when you change V and W they span the space the space. So, this certainly means, so let us just give this space set a name script B, B certainly spans the vector space V tensor W over F. Now, we just need to show that B is linearly independent.

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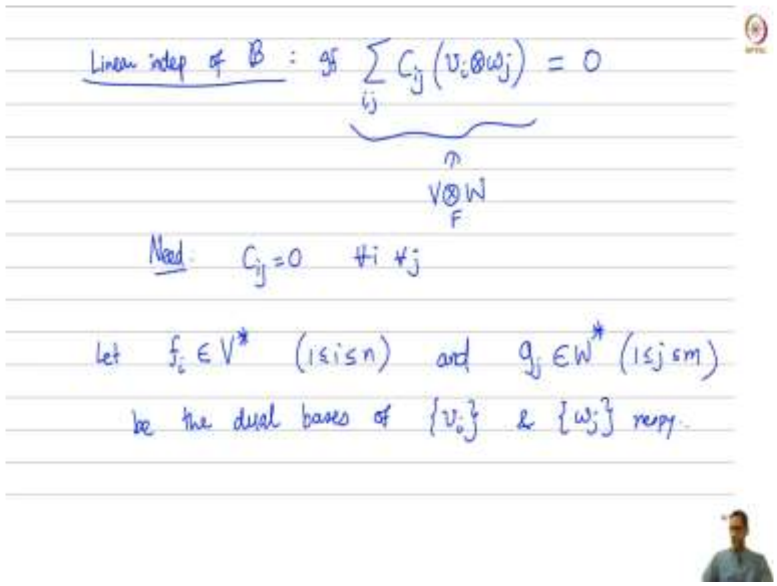

So, let us show linear independence of B. So, what that means is you need to take a linear combination Cij vi tensor wj, so this overall I's and j's running over the appropriate indices suppose this is 0, now what is this? This is an element of V tensor W, so I am saying suppose as an element of V tensor W this sum is 0, then I need to show that this element itself is, I am sorry, I had to show that all the Cij's are 0, so I need to show that Cij is 0 for all i and for all j.

Now, to do this requires something a little bit more, we need to work slightly harder, so observe that when I have basis vi and wj of my vector space, I also have what are called dual basis, so let us give those also names, let fi belong to V star this is for i from 1 to n and gj belonging to W star be the dual basis, corresponding to the v's and the w's, respectively. Now, what does dual basis meaning?

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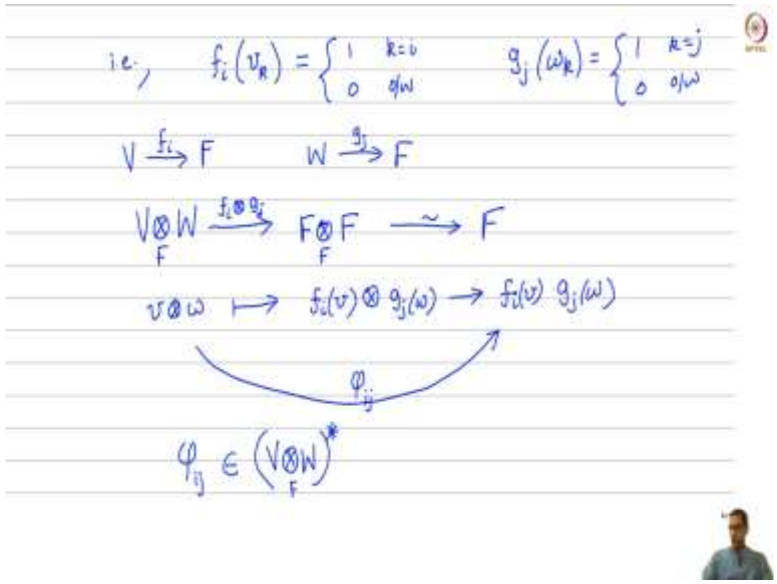

Recall that these dual basis elements are just following when I evaluate fi on a given vk, then it gives me 1, if k is i and gives me 0 otherwise. Similarly, if I evaluate gj on wk it only give me 1 if k is j, otherwise. So, these are what you call the dual basis. Now, observe that fi and gj are linear functional on V so in fact what that means is that I can think of it as follows, it is a map from V to F, this is what fi is, gj is a map from W to F.

So, now let us use something that we learned about tensor products that this tensor is really a functor, in other words not only does this give me a map, I mean not only am I constructing an object called the tensor product of two vector spaces, if I have maps between the components from V to a vector space and W to a vector space, then it induces a unique map from V tensor W to the tensor product of those two vector spaces.

So, if I take this map here, this is fi what we called as fi tensor gj, so this is a map from V tensor W over F to F tensor W over F. Now, what is this map do recall, it takes a simple tensor of the form. V tensor W and maps it to fi of v tensor gj of w. But now recall, we have just now done this the previous problem says that F tensor F is actually just F. And what was that map? It was just the product map. So, this is fi of v into gj of w. Now, what is this mean? It says that I have a map from V tensor W to F we just takes V tensor W and maps it to fi v time's gj w.

So, this is a well-defined there exists a well-defined map, so maybe we should just call this composition as something, maybe I will call it phi sub ij. So, phi ij is therefore a linear functional if you wish. So, what is phi ij? It is a linear map F linear map from V tensor W to F. In other words, it is an element of the dual space. So, what we have constructed is an element of V tensor W star. Now, why is that useful? Because this is what we were trying to prove, we have given summation Cij wi, tensor vi tensor wj is 0 and from this we need to somehow how that all the Cij's are 0.

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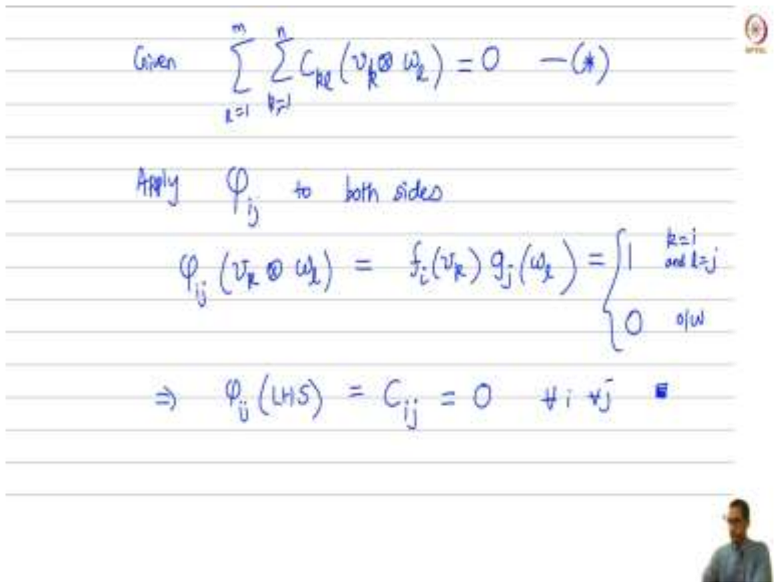

So, now let us do. So, you given this, so maybe we should just change these indices for the moment. So, let us just call it Ckl vk tensor wl, so the sum is over all k going from 1 to n, all l going from 1 to n. Now, what we do is we will apply this special element phi ij that we just constructed to both sides of this equation.

Now, what does phi ij do? Well, when it acts on vk tensor wl, look back on what it did, it is just fi of vk multiplied by gj on wl. And since f and g are just the dual basis, this will give you 1 or 0, this gives you 1 precisely when k equals i and l equals j and it give you 0 and all other cases for all other pairs. So, what does that mean? Which implies when I apply phi ij to the left hand side phi ij of the left-hand side of this equation here all the terms are going to become 0, except a single term which corresponds to vi tensor wj.

In other words, I will just get Cij as my answer. And of course the right hand phi ij of the right-hand side is 0. So, what I have done is shown that phi ij of the LHS which is Cij is 0 and this is true for all i and all j, I can just change my i and j and so I am done that is exactly what I wanted to prove. Let us go on to the next problem.

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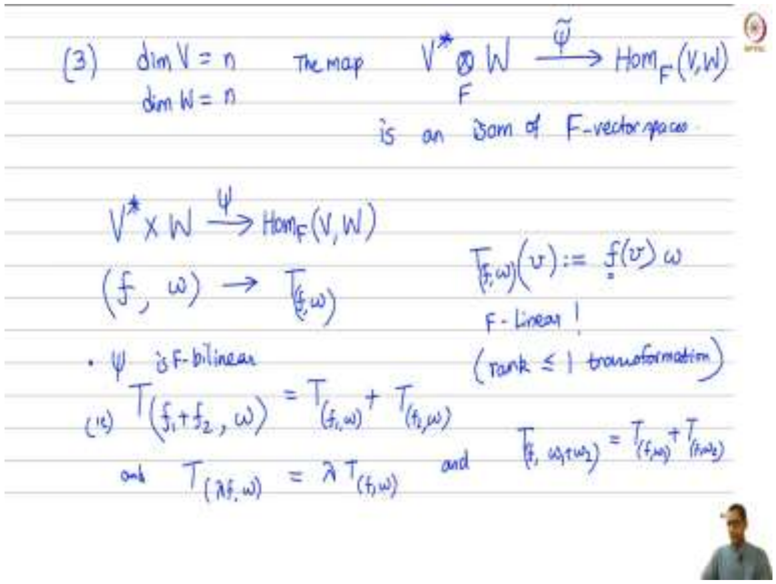

Now, suppose I have the same hypotheses that V and W are finite dimensional vector spaces. Let us, say this is dimension n and this is dimension m, then I claim that there is a certain natural isomorphism between the following two vector spaces, from V star tensor W to the space of all linear transformations, what I will call as Hom F linear maps from V to W, then there exists a certain map then so let us call it psi then the map psi which I will define in a moment is an isomorphism that is a claim.

So, what is the map itself? Well, the map is of following, so first if there exists a map as follows, which is so firstly maybe we should start one step earlier which is we should first try to construct some bilinear maps, so let me give you an example of a bilinear map from V star cross W to the space of all homomorphism's. So, V star recall is the dual space to solve the linear functional's and W is just elements of W corresponding to this pair I will ask I will associate a certain linear transformation.

Now, what is this linear transformation? If I take T corresponding to the pair fw, when I evaluate it on a vector v, my definition is a following, I can evaluate f on v that will give me a number and then I can multiply the answer by w. So, this answer now is in w. So, observe that this definition is well firstly the output is certainly an element of W, this is a linear transformation because when I change W linearly, when I change V linearly, let me get v1 plus v2 or some scalar multiple of V then since F is linear, you know, this splits either as F of v1 plus v2 or as lambda time's fv.

So, this is in fact F, this is definitely F linear, because this linear functional F is after all F linear. Now, therefore I have certainly defined an operator in or a linear transformation in hom F V W and in fact if you think about it a little bit you will see that what sort of operator have we defined or what sort of transformation this is something which is of rank 1, this is this is a rank 1 transformation which the range of this transformation is exactly 1 or if maybe this linear function is 0, then it could just be the 0 operator.

So, to be very precise we should say, well, what sort of thing are we doing here corresponding to a linear functional F and a vector W, we are associating a rank at most 1 linear transformation. Now, observe that this map is bilinear, so this association that I have here, so we will call this psi tilde and call this bilinear map as psi, so the other thing is that psi itself is bilinear, is F bilinear, ie, what should we check if I change f to f1 plus f2 and keep w the same, then what I get on the other side is a some of the answers, so in other words T, so I should have put that this t of f sub the order pair fw.

So, I claim that the operated T f1 plus f2 w is just the sum of the operators, T f1 w plus T f2 w. And similarly, for scaling and so on. So, if I multiply this by some scalar T of lambda f w, it is just lambda T of fw. And I should do the same thing in the other component, which is T I keep f constant I write this as w1 plus w2, then this just becomes T is f w1 plus T f w2.

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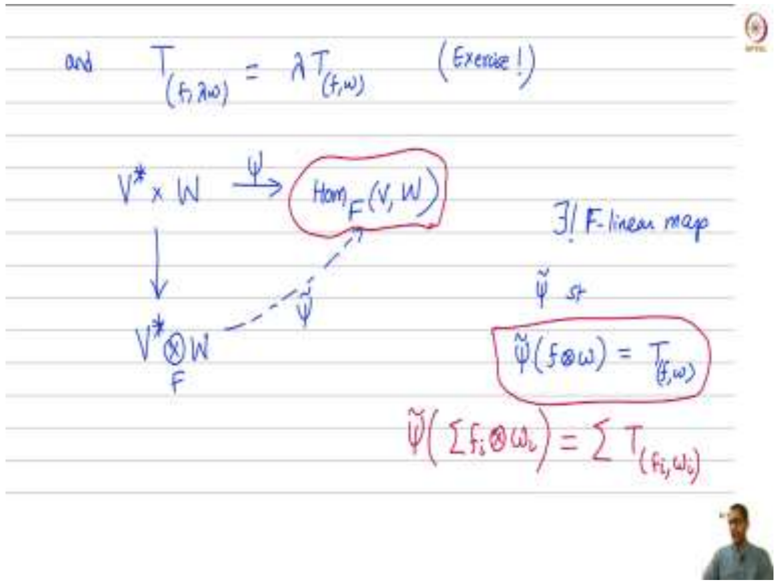

And finally the last thing if I change this to a lambda then the lambda come out. So, I am going to leave all these 4 verifications as an exercise, it is just straightforward application of the definitions, so what this means for us is that you have this nice way of combining a map like this, which is an element of V star cross W, I mean you have given a linear functional and a vector there is a nice way of constructing in a bilinear session a rank 1 transformation.

But the important thing is that this bilinear. Now, what does that tell us? It tells us by the universal property that I have a map from V star are cross W take all psi to the space of homomorphism’s and now by the universal property I know that there exists a map unique map psi tilde which makes this diagram commute. So, there exists a unique F linear map psi tilde, such that psi tilde on what on f tensor w is the operated T fw that we associated.

Now, the claim is that this map is an isomorphism, so that is my point, finally, the map psi tilde defined in this way is in fact an isomorphism of vector spaces. So, observe that firstly it is a on two map that is again the easy part of the proof.

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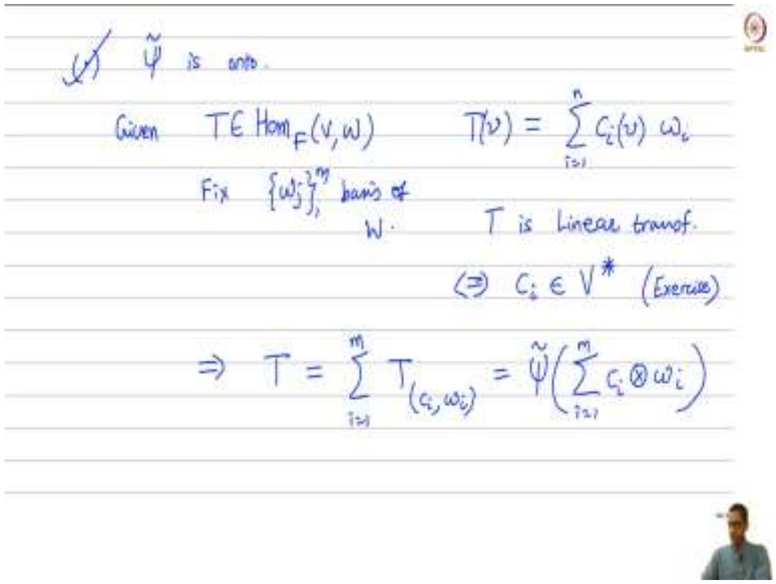

Observed that psi tilde is definitely onto, why is it onto? Well, because if you what should you show to show that it is onto? You have to take an arbitrary linear transformation, here take some random linear transformation and show that you can obtain that in the image of psi tilde, psi tilde is this, this is what psi tilde does. Now, how do I obtain any linear operator as the image of something psi tilde?

So, the point is this is only the action of psi tilde on the decomposable tensors, remember, so more generally if I write it as a summation, then it will become the corresponding sum, so psi tilde on a sum of the following form fi tensor wi is therefore the sum of T of fi wi, again that is the key. Now, the point is, any linear operator linear transmission can always be written as a sum of rank 1 transformations.

So, let us just see why that is true, so given any T, so what does T look like? T acts on V, it gives you some vector in w and let us fix a basis of w, let us fix a basis like we did before, fix a basis of w, so then since Tv is a vector in w, I can write it as some unique linear combination, in fact summation Ci Wi. But of course these Ci's these scalars depend on v, so I will indicate it like this, the constant Ci, the linear combination the scalars occurring there depend on V and Ci of v observed so T the key fact now is the following T is linear implies that the Ci's are actually linear functional's, they are also linear operators.

In fact and if and only if said, so if T is a linear transformation from V to W, then it means that these coefficients are all linear functional's. Again, easy exercise, just linearity of T translates into linearity of the ci's. So, what does that mean? Then that just tells you that by definition T therefore is just the sum of T, so the ci's are now linear functionals and the wi's are some vectors.

So, we have managed to write the given operator T as a linear combination of operators of the form T of ci wi. Therefore, what this means is that the given operator T is actually in the image of psi tilde, this is just it is just in its psi tilde acting on summation ci tensor wi. So, we have shown that psi is onto now the isomorphism bit is now just going to be from a dimension count.

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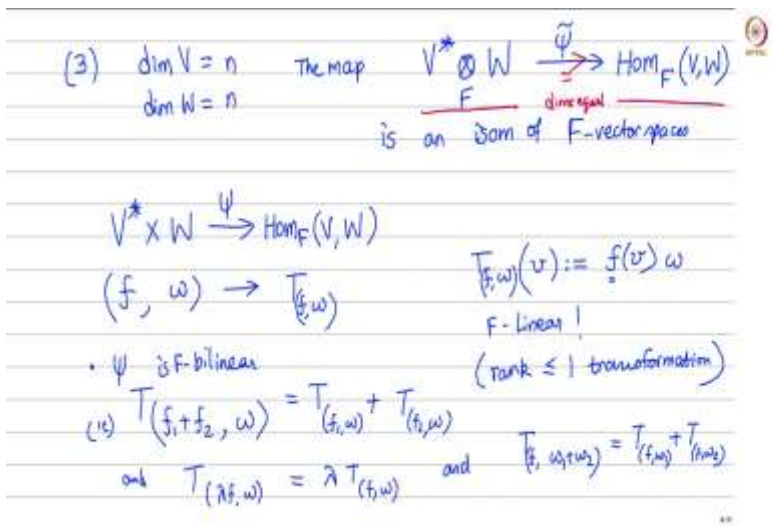

So, observe that we have a vector space here, a vector space here, you have a onto map between them. Now, if the dimensions are equal, then that onto map must also be an isomorphism. So, let us calculate the dimensions on both sides.

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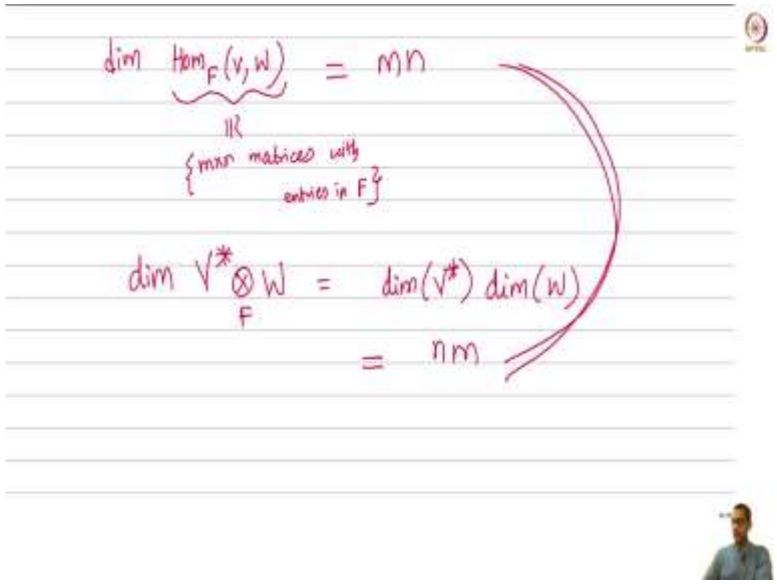

So, if you see what the dimension of Hom F VW is, that is just like, so what are these? These are linear transformations, so what is the dimension of home F VW? This is like you can think of this as linear operators, this is like n cross m matrices if you wish or m cross n matrices, you can identify this with all, you know if you think of matrices with entries in F and you can identify it with that vector space, just by choosing basis on both sides.

So, what is that? Linear operators are really matrices and matrices are well m cross n matrices, this have dimension mn. On the other hand, the dimension of V star tensor W as we just did in the previous problem, is the product of the dimensions, so this is dimension of V star times the dimension of W, but the dimension of V star is the same as the dimension of V. So, this is again nm, which means that these two dimensions are equal and we are done. Because then going back and onto map here is a onto map and the two dimensions are equal, dimensions equals, therefore this onto map must also be an isomorphism.

So, this is an important isomorphism, this gives you a very natural way of thinking about what matrices are, they are also like a tensor product. And like we said before, when we talked about functors and so on, this is a natural isomorphism that somehow the key point here that you can say you can make this completely functoreal and so on. But you know we would not let us not bother about that right now, let us just think of it as given any V and W there is there is such an isomorphism.