Algebra – II
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Extension of scalars

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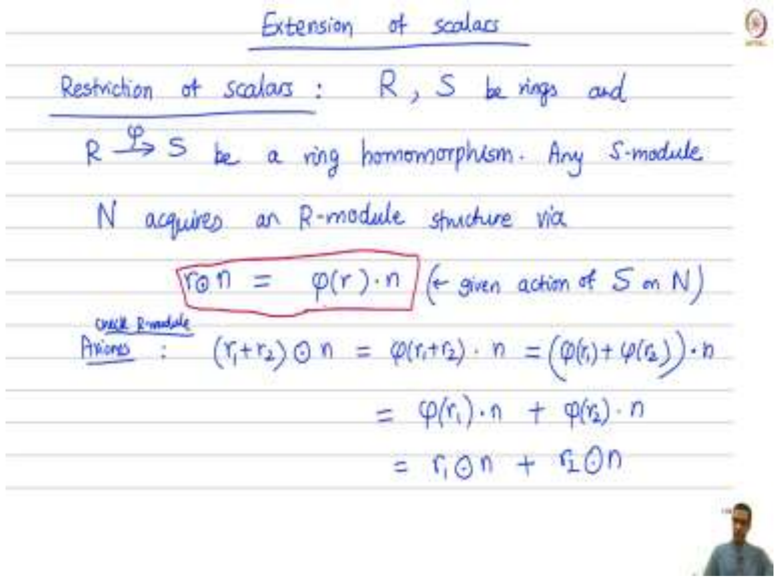

Let us, look at some special cases of tensor products which are very important in their own right, so here is one application, this is something called extension of scalars. So, here is the general problem, so first there is let me talk about the dual notion something called restriction of scalars. So, what is this? What does this mean? So, suppose I have two rings, so let R, S be rings and let phi be a map from R to S, be your ring homomorphism.

Now, what is this? What is this do? Any S module automatically becomes an R module, any S module N acquires and R module structure it becomes an R module, so all this is left modules and R module structure, so here I am not talking about commutative rings necessarily, it acquires an R module structure via the following definition, I want to make r act on an element n of N, well what I do is I first act phi to r, so this is known element of s and I am of course given that n is an s module.

So, then I just act on n, so this is the this action now on the right hand side is the given action, the scalar multiplication of S on N, so when I say action I mean the scalar multiplication. So, this is my new definition, how am I going to make an element of R act on N? I will first convert that element into an element of S via this homomorphism phi and then make that phi of R act on N. So, this becomes an R module, well let us check the axioms of an R module.

So, axioms let us check, R module axioms. So, what all do we need to check? Well, N is still a abelian group that is all right we just need to check things like if I take r1 plus r2 and act on n, will it give me the sum, is there distributive T? So, this by definition is phi of r1 plus r2 acting via the given action on n, but phi is a ring homomorphism, so this is phi r1 plus phi r2 acting on n.

And now by the fact that the n is an s module there I have distributivity, so this is phi r1 n plus phi r2 n. So, this by definition is r1 acting on n plus r2 acting on, so I proved the first property. Now, so two things one is on the other hand it is a on the right hand side it is a module over s, so those axioms are satisfied and phi is a homomorphism of rings, so in some sense everything will carry through similarly.

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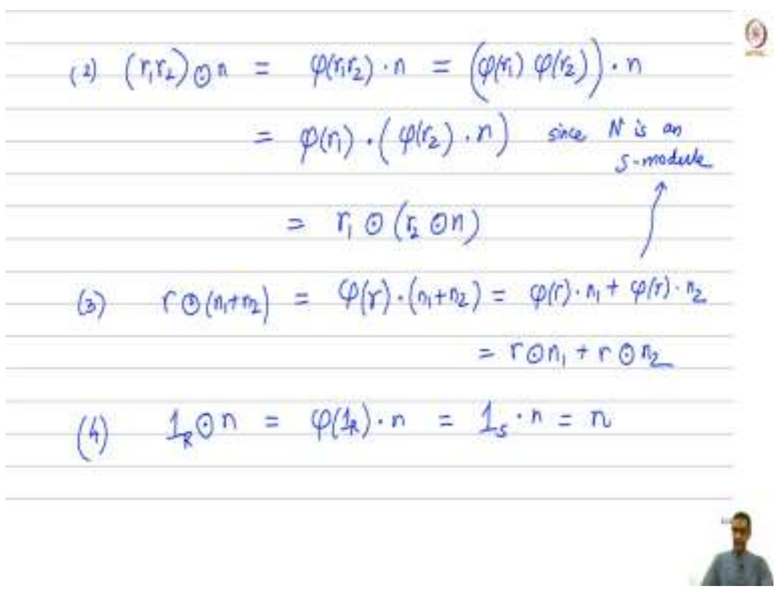

So, property to for example is r1 or r2 acting on n, what is that? It is phi of r1 r2 which is the product because phi is a homomorphism, how does the product act? Since N is an S module and that is what we need because this is now r1 acting on r2 acting on it, axiom 3 says if I take r1 acting on n1 plus n2 or r acting on n1 plus n2, so this is phi of r, it is phi of r usual action on n1 plus n2 but then this has this satisfies this axiom phi rn1 plus phi rn2.

Again because n is an again because of this reason that is rn1 and finally the identity we take the identity element the multiplicative identity of r, how does it act on n? Well, I first apply phi to it but homomorphism's necessarily map the multiplicative identity to the multiplicative identity and identity on n is N. So, we have checked all 4 axioms. So, this is what is called the restriction of scalars. Now, so what are the typical example?

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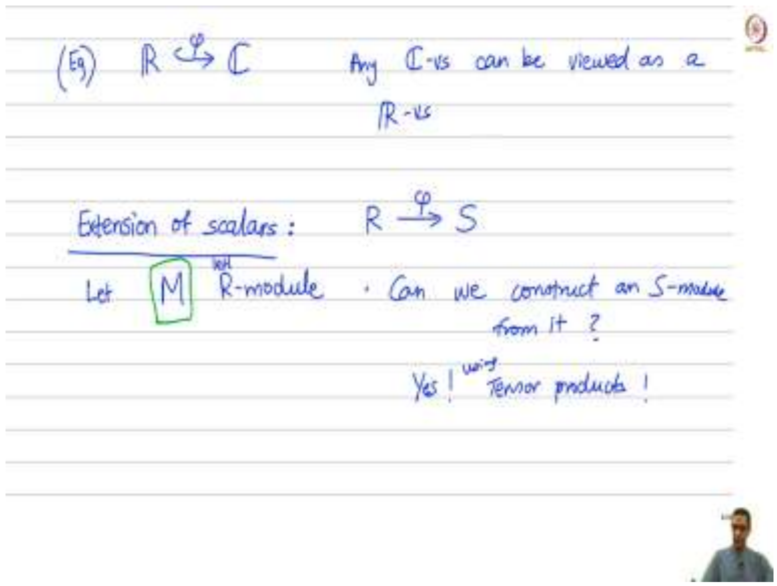

So, for example if I had a vector space over the complex numbers can always be thought of as a vector space over the real numbers, why is that? Because the ring well in this case a field, the field of real numbers has an inclusion map, this is my homomorphism phi, just an inclusion homomorphism.

So, the real numbers has a homomorphism to the complex numbers, so any complex vector space automatically becomes a real vector space, can be viewed as a real vector space by restriction of scalars that is exactly the same definition here it is just the inclusion map and the inclusion map is exactly what corresponds more generally to restriction of scalars via arbitrary homomorphism.

Observed because many things change if I have a vector space of some dimension, for example the complex numbers itself as a complex vector space its of dimension 1, but as a real vector spaces of dimension 2, so this is a new structure in some sense, if you wish. Now, extension of scalars is sort of the opposite problem, so let us pose the problem here, extension of scalars, so same thing I am given a ring r and m given a ring s the question now is, suppose I have an R module, so let M be an R module, the question is can one construct an S module from it?

This is like asking I have a real vector space can I somehow construct a complex vector space out of it? So, it is sort of a vaguely post question at the moment, but broadly the idea is this, can you somehow going the opposite direction? So, the same thing that we tried earlier is not going to work anymore because I only know how to make elements of R act as scalar multiplication.

If a given an element of S, I do not have a way of converting into get into an element of R, because the map phi only goes in this forward direction, I cannot apply phi inverse there is no such thing as phi inverse, phi need not be invertible for example. So, the point is you cannot quite do the same thing as before, but it turns out that one can never trust construct a useful notion of an S module which arises from the given R module M. And to do this we will use tensor products. So, the answer is well there is a useful notion and we will use tensor products. So, let us go to the construction now and see how this is done.

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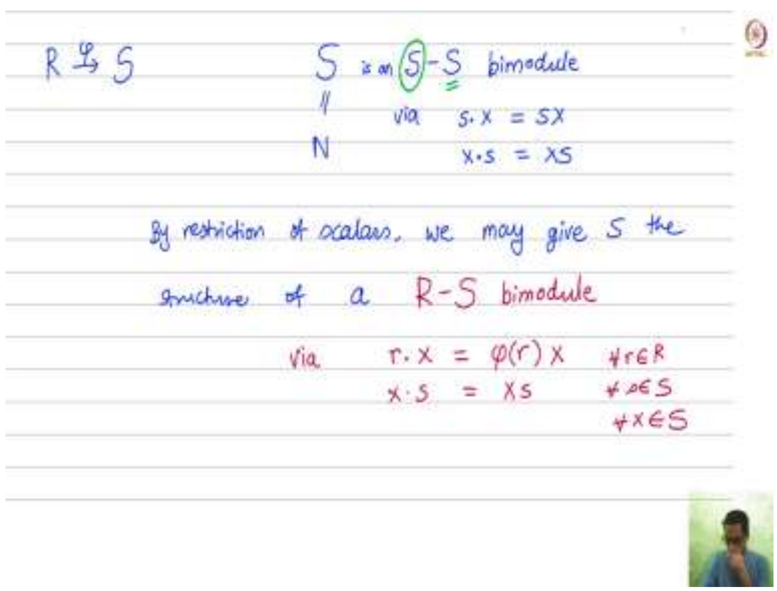

So, again for this we need this whole by modules business that we talked about. So, recall I have my ring so there is R ot S, this is a setup, the ring S itself recall can be thought of as a S-S bimodule. How was that? You have left multiplication by S and right multiplication by S. So, via this is just multiplication in the ring. Now, the point is because you have this now there is remember we know how to do restriction of scalars.

So, here is imagine this is like your module N, N is a left S module, it is also a right S module, now what I do is I do restriction of scalars to this, what I mean is I will now by restriction of scalars we may give S the structure of, well what can I do? So, let us just look at this, forget the second S on the right, so look at this S, S is a left S module, so by restriction of scalars I can think of it as a left R module.

So, and then I will put back this S, so here is what I can do, I can make it into a left R module and keep the same S module structure as before, I can make it into a R-S bimodule. What is the definition therefore, how should I make R act on the left? What does restriction of scalar say? You act phi you will get an element of S and you make that element act as it is. And how what is the right action? Well, that is as before that I am not changing. So, this is now for all r in R or s in S. So, this is how I make it into a right rather into a bimodule into an R-S bimodule.

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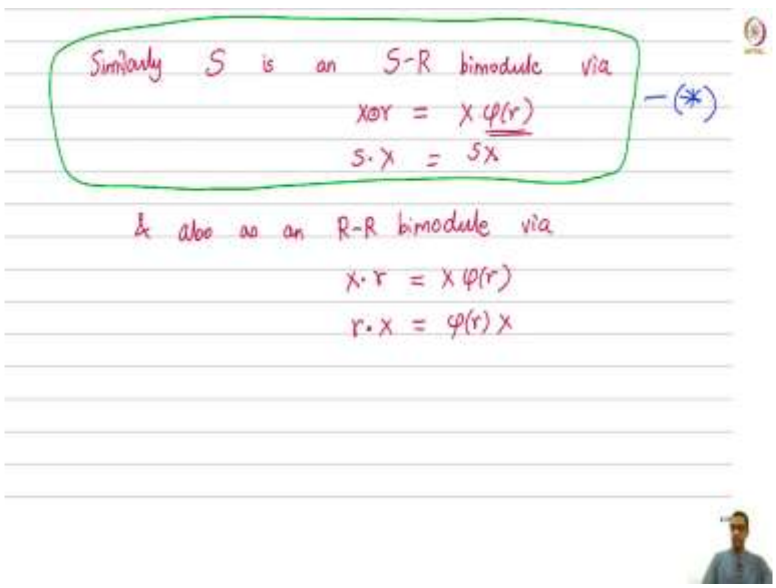

Now, not just that that is only one of the possibilities we can also make it into a similarly S is and S-R bimodule, we can do a similar thing, think of it as a S-R bimodule, here I am restricting the action of the right via how does R act on the right, S act on the left as before. So, on the what I have written here on the right sides are just the multiplications in S.

And also as an R-R module if you wish, also as an R-R bimodule, here I will change both of them x acting on r is x multiplied by phi r, if r has to act on the left, then I will just restrict the action as before, so this is just restriction of scalars. So, there are many different things you can do now, but let us focus for the moment on this one, you let us think of S as an S-R bimodule and what else was given? Let us, go up here, we had M, which is an R module. So, of course R module, of course means it is a left R module. So, now that means I can I can do something with tensor products.

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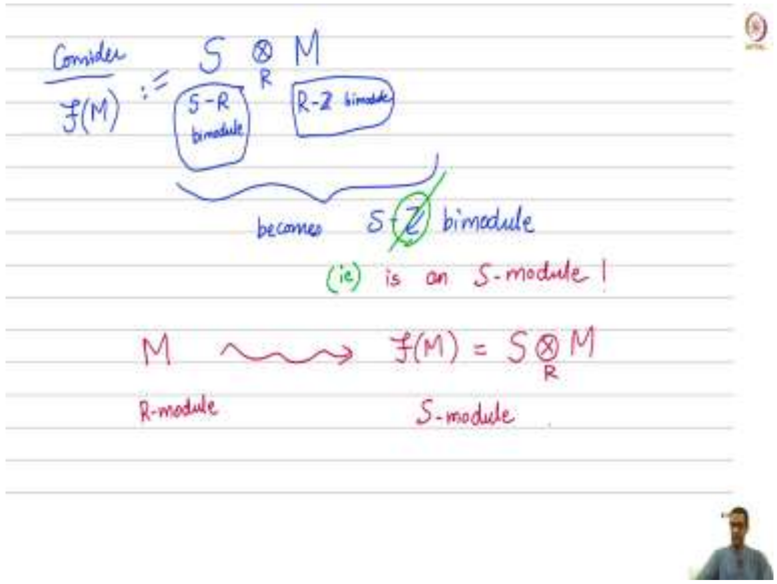

So, M recall is an R module, so iti s an R-Z bimodule if you wish or like ((\))(12:53) said this before, it is only a left R module, you do not need to worry about putting the extra Z there. Now, let us do the following, I am going to tensor M with S on the left. In order to do that, I have to view S as a bimodule, so I recall I said it is an R, well it is an S-R bimodule, I want to think of S as an S-R bimodule.

And I will think of M as an R module or if you wish as an R-Z bimodule, whichever way maybe we will just put bimodules for now, R-Z bimodule. So, consider this, consider this new object. Now, let us call this something, let us call this f of M, given an M I have constructed a new object S tensor M over R where this tensor product I make sense out of it like this S is because it is a ring with a homomorphism from R to itself, I think of S as an S-R bimodule via this equation star, this is my action.

Now, given an S-R bimodule and an R-Z bimodule, I know how to do how to make sense out of the tensor product. So, this final object is therefore, it is an S-Z bimodule becomes an S-Z bimodule by our general discussion of tensor products and bimodules. Now, this object is exactly what we mean by what we get by extension of scalars, because observed to say that it is an S-Z bimodule, I mean I can ignore the Z, the Z module structure does not add anything more, it is just the abelian group, so what I have therefore, ie, this is an S module is an S module.

So, this construction has done the following M on the one hand, I started out with an R module, a left R module and starting with m I constructed this new thing called f of M which is s tensor M over R and this new object is now an S module. And this guy is what we this construction is what we think of as extension of scalar starting with an R module how do you construct somehow a natural S module from it? So, this is we say that this is obtained f M is obtained by extension of scalars.

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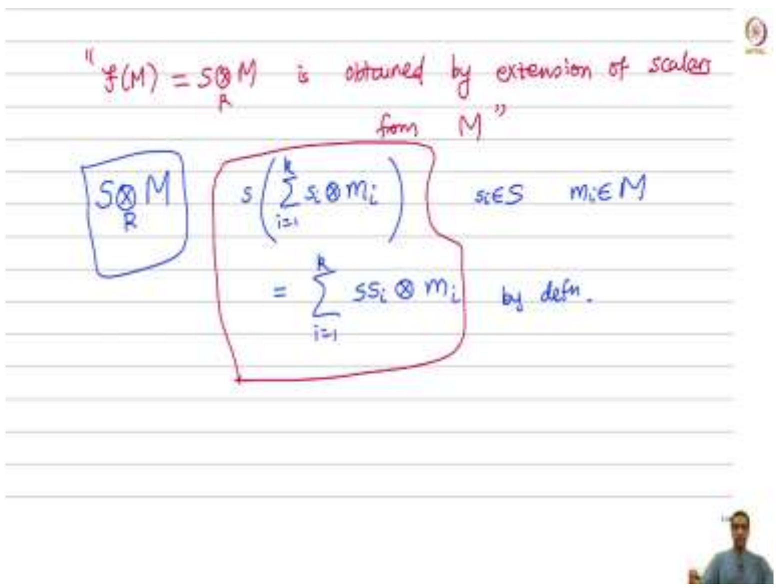

So, this is the usual way it is expressed f of M is obtained by extension of scalars from the module M. So, how what is the scalar multiplication also, let us just recall, how did this become an S module? So, S tensor. M, firstly what are the elements look like? Well, they look like summation si tensor mi, these are the typical elements now, finite sums of this kind, si is come from S, mi is come from M and again as before recall this is not a unique expression, so that may be many many ways of writing the same element as such a linear combination.

But what is scalar multiplication? So, recall if I take a scalar s from S and I try to act on it, so if we recall how this whole bimodules business worked with tensor products, this is just obtained by multiplying the first component by s, this was the definition. So, this is how scalar multiplication now works on this new module s tensor m. Now, what is the point of this? In what sense is this sort of the most natural module to which we want associate to N? What is the special property that it has? That is the key.

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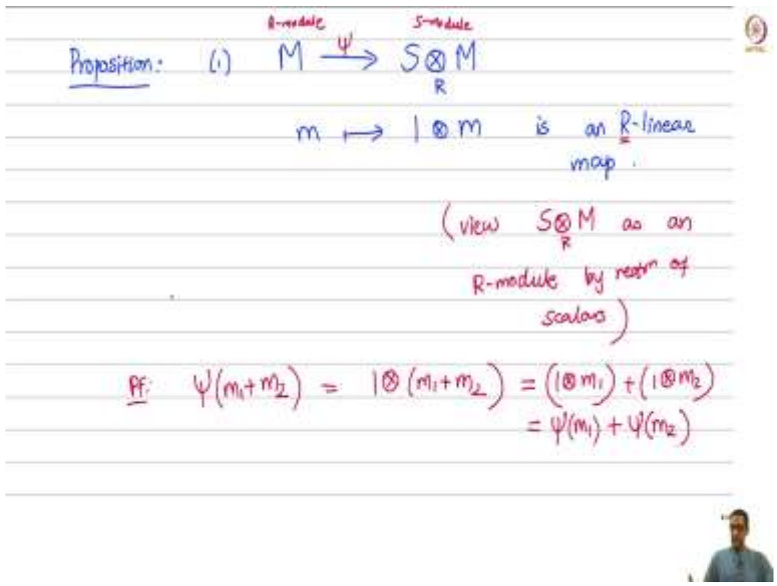

So, here are the key properties, well let us call it a proposition, this new module S tensor M has the following two properties, number 1, there is sort of this natural map from M to this new module, let us call it, remember there is a multiplicative identity in my ring S, so I can map each element m of M to S tensor M I mean, to 1 tensor M that particular element of a S tensile. So, observe this map, now the point is that this map what sort of map is this?

This is an R linear map, so recall on the left side, M is only an R module, so I can think of M as an R module, the right hand side recall was an S module, so when I say, so this guy is only this was an R module, the right side is an S module, but what am I saying here that this map let us called it something psi I claim that this is an R linear map, what do we mean by that?

Both sides have to be R modules for something to be for this to even make sense, what I mean by this is I view the right hand side view S tensor M as well it is an S module therefore an R module by restriction of scalars, as an R module by restriction of scalars that something it can always do. So, then I think of both as R modules and then I claim that this map psi is actually an R linear map.

So, let us prove this proposition first, part 1 of the proposition, so how do we show it is R linear? Well, I have to show if I take a sum of two things, what does it go to 1 tensor m1 plus m2 and of course this map this tensor symbol you can think of it as it is exact linear definitely, so 1 tensor m1, therefore it is psi m1 plus m2 that is the easy property.

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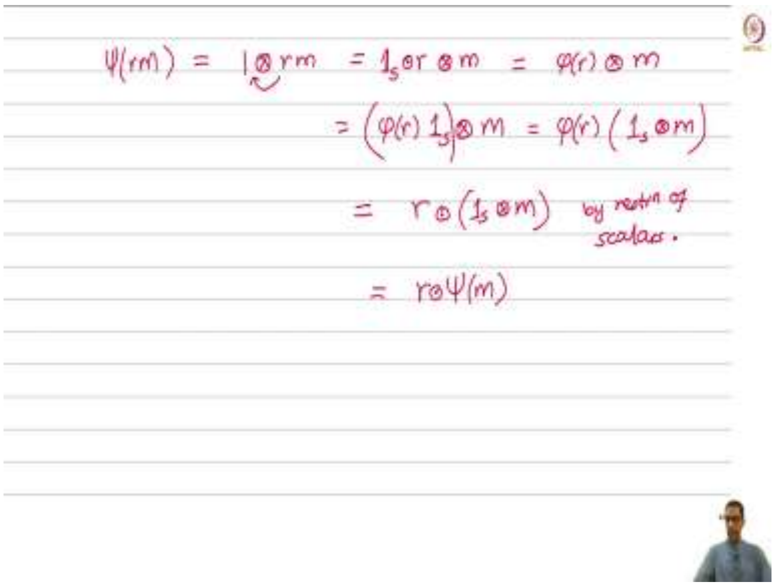

Let us, check the second one, which is that what happens if I put r in front of the m? What is psi of rm? Well it is 1 tensor rm by definition. Now, recall this tensor symbol here is also R balanced, because I am when I am doing this, I am sort of taking the universal property with respect to balanced maps, so if you go back and look what I can do is the r to the other side, so this is 1 times r which is r again, I had to be a little careful, so this is 1, 1 is maybe we should just be very careful with what belongs where this is the multiplicative identity of 1, right multiplied by the element r tensor m. So, this is the new write multiplication which I have.

Now, what is the right multiplication? If you go back and recall how we made S into S-R bimodule, the right multiplication is the following, you if you want right multiply by R you just multiply ordinarily x with phi r. So, I have to do that here. So, I should make this so x here is 1s, so when I multiplied by phi r just becomes phi r, so it is phi r tensor m.

Now, phi r we call, well I can think of phi r as 1 into phi r is phi r into 1 I can just put this 1s on the other side, so it is this element of s tensor m, but recall that is exactly how the S module structure is defined, how did we define the S module structure here? So, we said if you want to multiply some element by S, you just have to multiply just the first components of all the tensors by S.

So, here what this means is if I take phi r acting on 1s tensor m that is exactly I multiply only the first term brand by phi r. But observe this is exactly saying, so phi r is just the r action via restriction of scalars, so this is just the r action on the element 1s tensor m, by restriction of scalars. Let us, go back and see what that was. How did we define the restriction of scalars? This is the map. When you want to make r act on something, it is just first apply phi and then make that element of S act on it.

So, lots of different definitions are sort of intermeshing here, so I would recommend that you sort of try and do this very slowly for yourself, check every step and see what is being used in every step. So, that is exactly r psi of m, where the r action is the action by restriction of scalars. So, what this means is that this map from m to s tensor m which takes each m to 1 tensor m, is actually an R linear map. So, that is the first observation.

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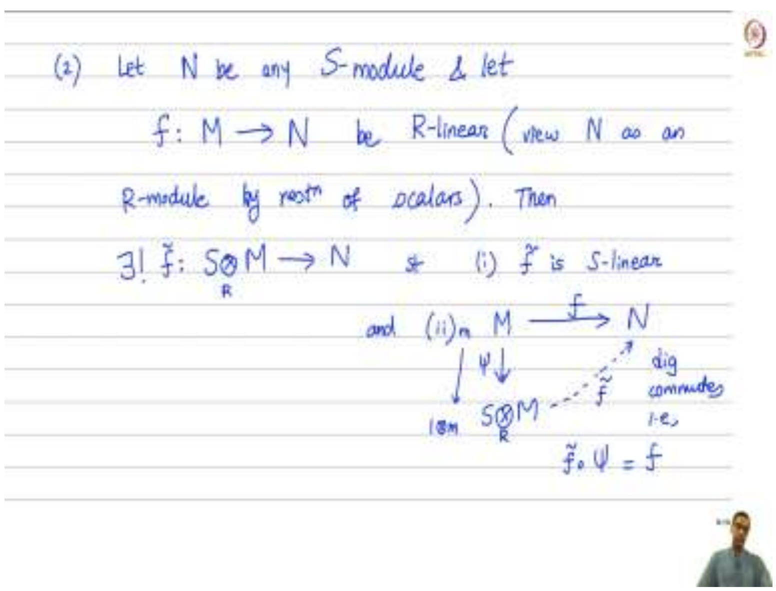

The second observation is that it is universal with respect to two such maps, in other words suppose I give you any let N be any S module and let f be a map from M to N, be R linear, again what do I mean by R linear? So, recall M is an R module, N is an S module, so when I say R linear what I mean is you view the right hand side, view N as an R module by restriction of scalars.

So, when you do this both sides become R modules and you can talk about R linearity, so f is an R linear map. Given any such map then there exists a unique map f tilde from well where is this map from? It is from the well it is a unique f tilde from the extension of scalars of M from S tensor M R to N such that two properties: 1, this map is S linear now, so now observe f tilde the right hand side is N, N is an S module, the left hand side is S tensor M and that is also an S module.

So, it makes sense to talk about S linearity here and so what we are claiming is you can this map f tilde will actually be S linear and it will make the diagram commute and to what diagram commutes? M going to S tensor M, so recall we had the standard map that we called psi which took every element M to 1 tensor M, then we have the given map f to N and we claim that there is a unique map f tilde like this.

So, such that this diagram commutes, in other words this diagram communes. And recall called diagram commutes just means that f tilde composition psi is the map f. So, the claim is that you can always find a unique f tilde which is now S linear and which makes this diagram commute. So, one way of phrasing this is to say that this extension of scalars construction is a way of converting R linear maps to S linear maps. So, let us just see why this is true. So, how would you do you construct this map?

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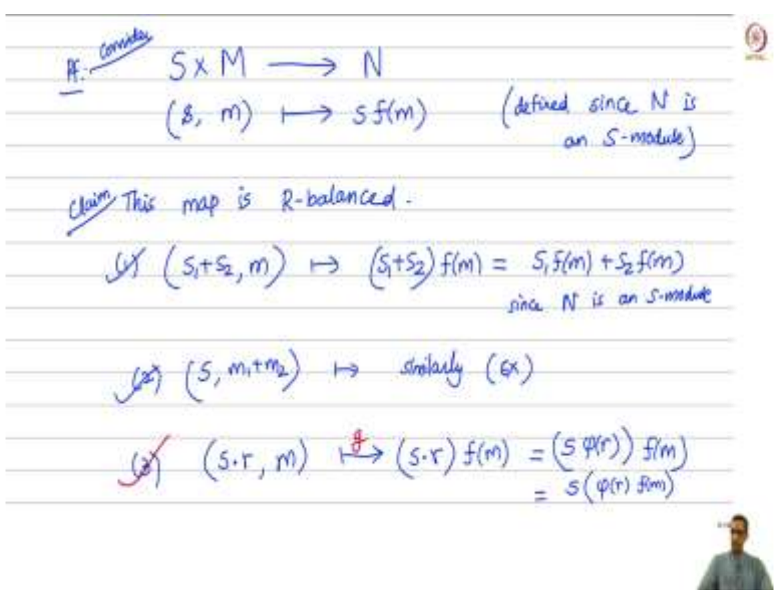

So, let us just use the universal property, so consider the following from S cross M to N we take an s, I take an m, so what will I do? So, I will just say s acting on f m, so this is well defined because remember N is a S module, so this is certainly defined, since N is an S module. So, this is a map, I claim that this map is actually R balanced this map.

Why is that? Well, property 1 is if I take s1 plus s2 comma m that goes to s1 plus s2 acting on fm, but that will split into two because it is an f module, it is an s module, since N is in S module the S module axiom and similarly the other, so this is all right and the same for s comma m1 plus m2, similarly so I will just leave that for you to check; exercise. And let us check the third and important axiom which is that if I take s and I acted on the right by r comma m, what does that map do?

That maps to well by definition its s acting on the right by r time's fm left multiplication, but how does r act on the s, how does r act on the right of s? So, recall that is just s phi r, this is again by definition acting on fm. Now, let us so then by the now they are all elements of s so I can think of this as s acting on phi are acting on fm.

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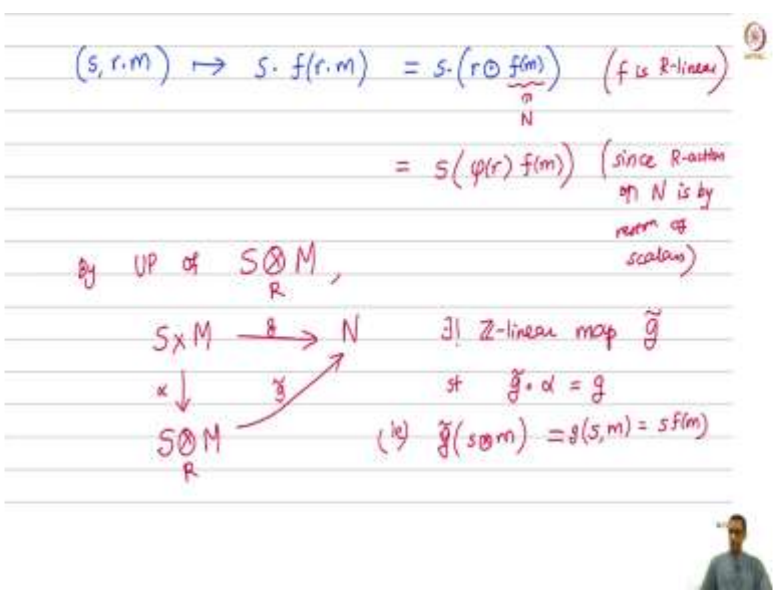

Let us, do the other side which is s comma rm, what is this map do under this map? So, by definition again this is s acting on f of rm. Now, what is f of rm? So, what we said is that f is R linear, so we had that property that f is R linear, so we can pull the r out that is what it means. So, this is s acting on r acting on fm. But how is the r now acting? So, we have got to be very careful, this is f because f is R linear, but now fm is an element of N which is only an S module, so how does r act on it?

Well, by restriction of scalars, so what does that mean? To make r act on something I first convert it to an element of s and then I act on it. Since R action on N is by restriction of scalars. And so that is the same answer as what we got earlier. So, the third property is also (())(30:29), so this map is in fact R balanced and so now by the universal property, so by the Universal Property of tensor product, what does it mean? It says that there exists a unique map S tensor M have a map to N, there exists a map S tensor M, so we did not give this map a name, maybe we should have done that.

So, let us call this this map g and so this is g and this map is g tilde. So, there exists a unique what sort of map? Well, this is a Z linear map that was the original universal product, g tilde universal property such that this is the map alpha g tilde composition alpha is g, ie, what does that mean? g tilde acting on an element of from s tensor m will just give you g of that, which is so it is g of s comma m, which is s acting on fm.

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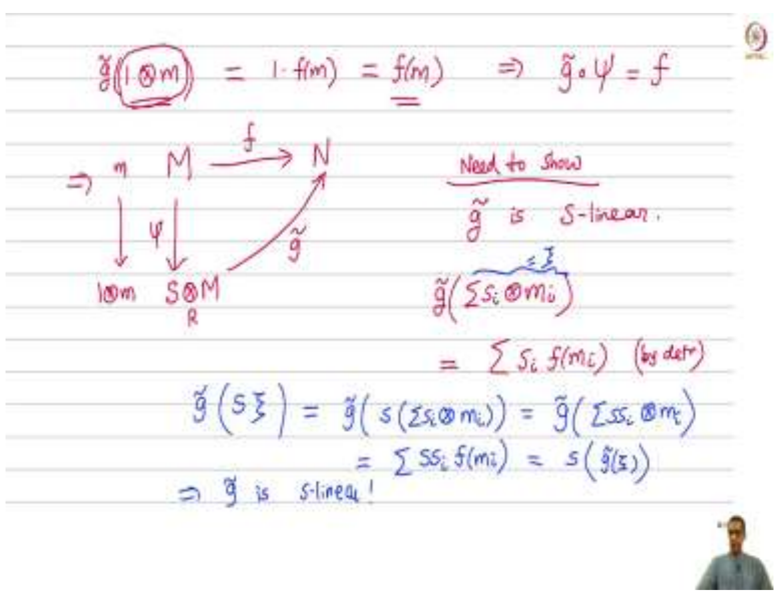

In particular, it means that g tilde acting on 1 tensor n is just 1 acting on fm, which is fm itself. So, g tilde is really the map that we want, so because g tilde certainly has the correct property that g tilde when it acts on 1 tensor m it gives me fm, which means I know form a different diagram, if you wish, so I have M and I look at my maps psi, M going to 1 tensor m, so I have argued that there exists a map g tilde f.

So, observe g tilde certainly makes this diagram commute. So, g tilde composition psi is in fact f, because it maps 1 tensor m to f of m. And g tilde is definitely, it is certainly unique this property uniquely defines g tilde that is because of this universal property, from here we know there is there can only be one map which does this for all S and M.

So, now so what do we have to show? We have to show that g tilde is in fact S linear that is the last fact. So, need to show g tilde is S linear. So, let us prove that how does g tilde work? What was g tilde? It was a Z linear map and in fact we know what g tilde does to a general S tensor M, it maps it to s f of m. So, now observe how does g tilde act on summation si tensor mi, by definition is summation si f of mi, this is how it is defined. So, this is by definition of g tilde.

Now, let us check what happens if you sort of multiply this by N S, so if I multiply this so this is my elements xi, so if I multiply this by s, so what is g tilde? On S xi can I pull the S out is the question. By definition, this is summation ssi of mi, now observe SSi f of mi just means that you know, I can just pull this you know, how so, so I have used something here, so let us back up a little bit, so S xi is by definition g tilde evaluated on S times the summation but there we are using the S module structure on this tensor product, which is that you just multiply on the first component. So, this is just summation SSi f of mi.

And now this just means I can pull the S out, so it is just S times whatever is left and whatever is left was just g tilde of xi, so g tilde is in fact S linear. So, we managed to show that this g tilde satisfies the property that we want, which is that let us go back up there exists a, so your f tilde is really your g tilde.

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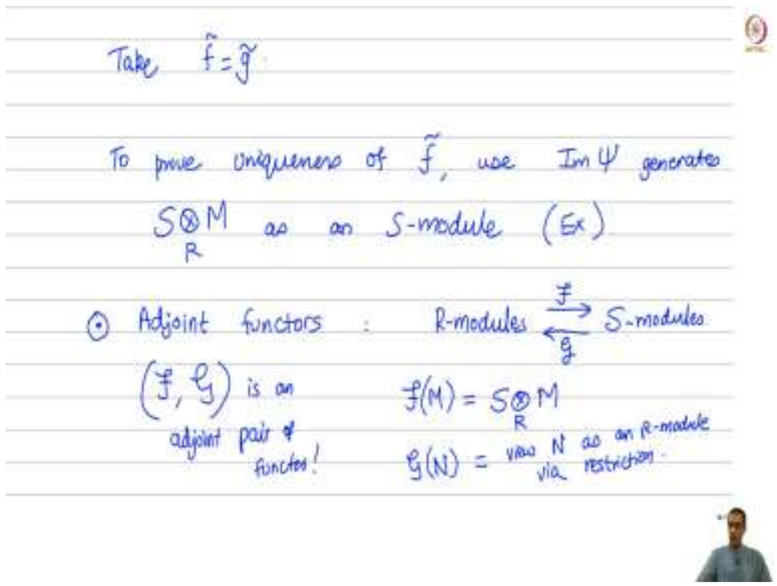

So, let us take f tilde to be g tilde that is end of the proof. Now, observe that what we have shown is that there exists at least there exists a map f tilde which does the job, why should this be unique? Well, there still a little bit more to prove that this is in fact unique, let us go back here that there exists f tilde which makes this diagram commute that is what we have shown and we have shown f tilde is S linear, what still remains to be proven is that such a map is unique.

But I am just going to leave that as an exercise it is the same sort of thing that we have done before then we prove the various universal properties, the image of xi, all elements of this form, they generate this as an S module, so for that is the last thing that is left, it is again analogous is to this, so for to prove uniqueness tilde use the fact that the image of this maps xi generate as an S module. So, prove this exercise.

Now, once that is in place then it automatically implies that this is unique. Now, this is the sense in which really the this is an extension of scalars, it sort of the most I mean it is exactly the object you will construct such that every R linear map from the original module M to some S module factor is uniquely through this extension of scalars module. And all of this can sort of be you know, let me just make a couple of concluding remarks here. So, you have looked at adjoint functors and so on.

So, all of this can be phrased beautifully in terms of the nomenclature I mean in terms of the framework of the adjoint functors and so on. So, there are two functors here the restriction functor and the adjoint the extension functor, so without going into too many details on the one hand, I have the category of R modules and like I said there is the category of S modules and actually there are two functors now, one is the extension of scalars functor that we just defined called f.

The other is the restriction of scalars functor f, g, so what is f? f takes and module M (())(38:37). And then I mean this is what it does to objects you have to similarly define it on arrows and the other side the g is the restriction functor, G N is you just view N as an R module via restriction of scalars, via restrictions, so there is the extension functor on the restriction functor and it turns out that these two guys are actually an adjoint pair, the extension comma restriction is an adjoint pair of functors.

So, they are both functors and further they form an adjoint pair. So, this is something you might want to try and prove for yourself that is really what the previous proposition says that it exactly meets the definition of an adjoint pair, that is one.

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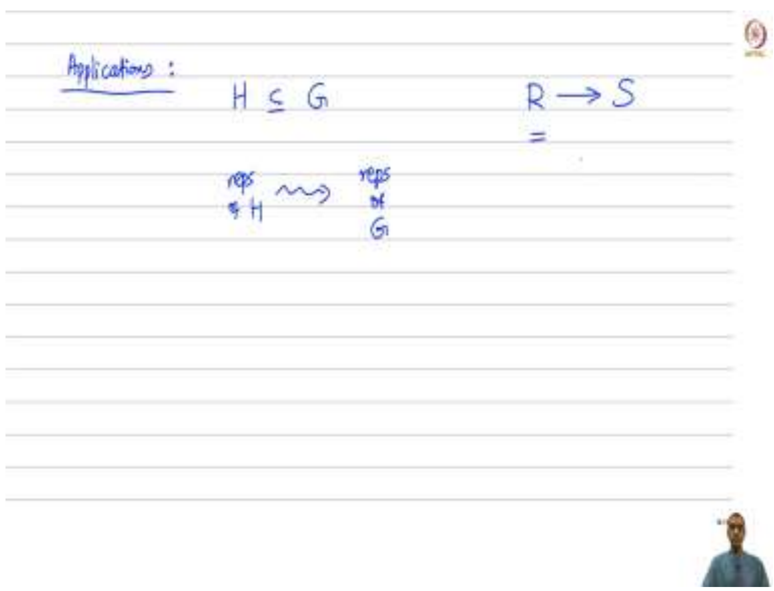

The other remark I wanted to make, is that, so this is this particular notion of extension of scalars is an extremely important notion, it is not just I mean this has lots and lots of applications, so I mean it is not trying to construct a real vector space trying to make a complex, trying to make a real vector space into a complex vector space in some sense, trying to define an action of the complex numbers that is of course one that for example, occurs in linear algebra and several places, but this sort of thing appears in finite group representation theory and so on, where often you have a subgroup H of a bigger group G and you somehow want to study what are called modules or representations of H.

So, you have representations of the smaller guy and from those you want to construct representations of the larger crew. And representations are like modules think of it as modules. So, this is a very important sort of thing because the smaller object is often easier to understand and from that you want to construct appropriate modules for the larger object. So, it is the analog of saying, I understand the ring R well and I know how to construct modules over the ring r, can I somehow use those modules to construct at least some nice modules for the ring S.

So, this is a very important sort of notion and of course what we have done is really used all the things we have done till now, we know things like bimodules and so on. So, it is the definitions are, they are all intermeshed and somewhat subtle, so you will probably have to work through this many times to ensure that you understand how every single equality comes about in in every one of those equations. So, I would encourage you to actually try and fight with this and ensure that you understand all the steps in the arguments