Algebra – II
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Lecture 67
Tensor Products of Modules over Commutative Rings

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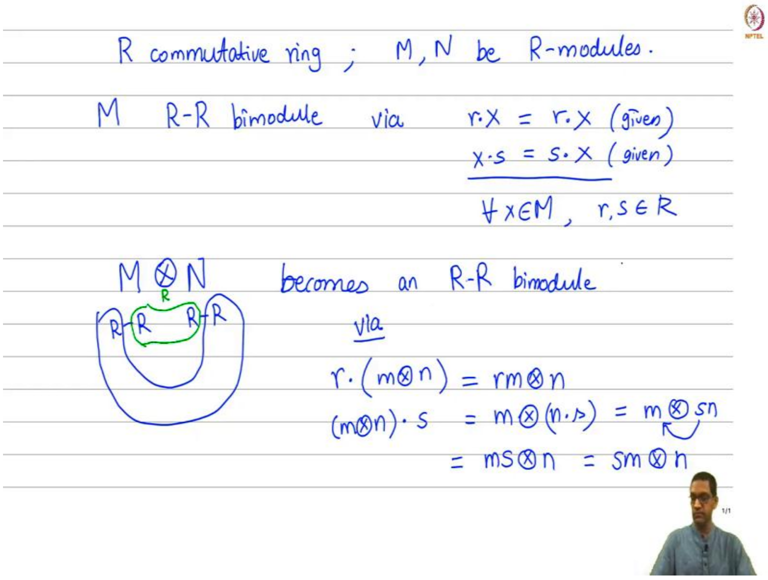

Let us, consider the special case of tensor products of module over commutative rings. So, I am going to assume now that R is commutative ring and let M and N be modules over R, so recall this important thing we talked about before, when we talked about bimodules, when you have a module over a commutative ring you can think of M as both the left as well as a right module over R, but also has a bimodule over R.

So, in fact, M as well as N are both bimodules, it is actually an R-R bimodule, via well what is the bimodule structure? It is just the same action so left multiplication action I mean action on the left is the given action, action on the right is just the same as a given action just think of it as multiplying on the left. And we had seen this before that because R is commutative this makes m into a R-R bimodule.

Now, the point is the tensor product as we discussed last time if M is an R-R bimodule and N is an R-R by module this means that the tensor product becomes again by the same token this R so this is tensor product over R, so this R and this R sort of are used up when we construct the tensor product and what is left is really this, so it is an R-R bimodule becomes an R-R bimodule, R-R bimodule and what is the structure or scalar multiplication here?

So, here is what we know if I take an element from r then the left multiplication, so when it acts on a generator or a decomposable tensor m tensor n, r acting on m tensor n by definition is rm tensor n that is the left action if you wish, the right action similarly m tensor n acting on s well by definition is m tensor n acted upon by s on the right, but as we just define for commutative rings the right and the left actions are the same. So, we this is just m tensor sn.

But the point now is that this tensor product operation is R balanced which means that if have a scalar on that side, I can sort of bring it pass the tensor product to the other side. So, what this gives me is just this is just ms tensor n. So, this is another way of thinking about it, but ms is the same remember as sm, because that is the definition tensor n.

In other words, what this says is that, so maybe if I replace the s by r everywhere, so observe I mean if you put r equals s then these two actions are actually the same, in other words, right multiplication by r and left multiplication by r on the tensor product is actually given by the same thing, which is you know, you can think of it either way. So, even though we say it is an R-R bimodule in some sense these two structures coincide, so it is only an R module if you wish, it became an R-R bimodule.

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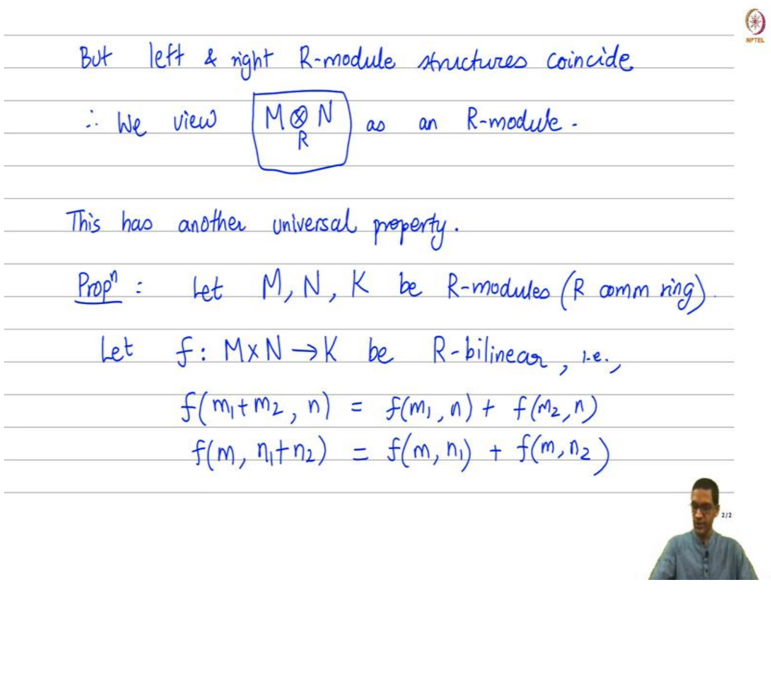

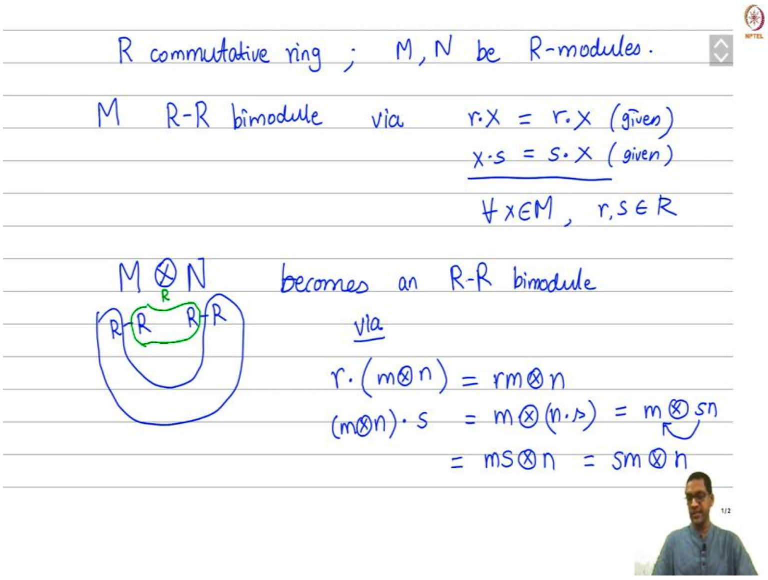

But left and right R module structures coincide and that is what we have seen above, you put R equals S, so therefore we view it only as an R module, so we view M tensor N as it is enough to say it is an R module, you can think of either of the two formulas as giving you the scalar multiplication think of it either as rm tensor n or as m tensor rn. Now, what more can we can we say? So, in this context, when R is a commutative ring, the tensor product not just does it have the defining universal property, so recall we more or less defined the tensor product as Z module with the certain universal property and so on.

In this case, there exists another universal property, so this has another universal property which is just sort of more useful some sense when we are talking about commutative rings and this is given by the following proposition, so let M, N and K maybe be R modules, so R is commutative and let f from M cross N to p, M cross N to K be R bilinear, now what is this notion of R bilinear mean? So, we have only talked about so far of Z bi-linearity and the notion of R balanced maps. This is sort of a little bit more than those notions, so what is R bilinear mean i.e., you say f of well it Z bilinear, so F of m1 plus m2 comma n is and the other way around m comma n1 plus n2, this is true for all ms in m and ns in n.

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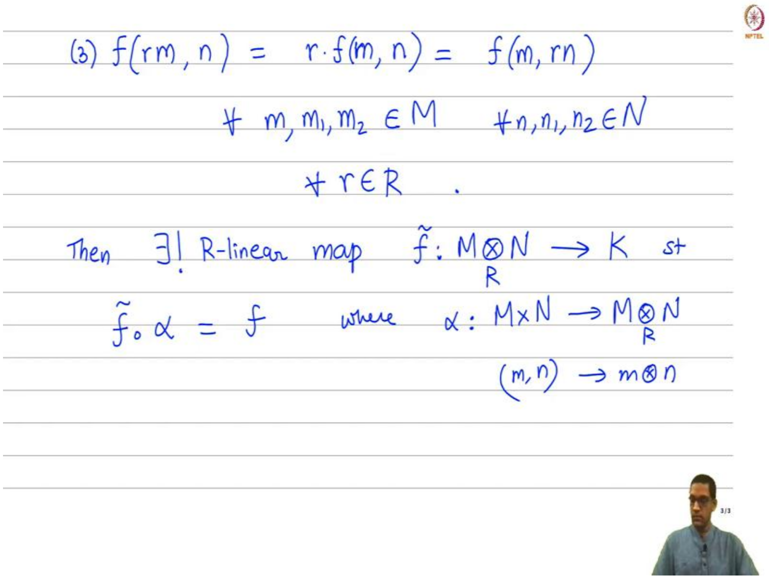

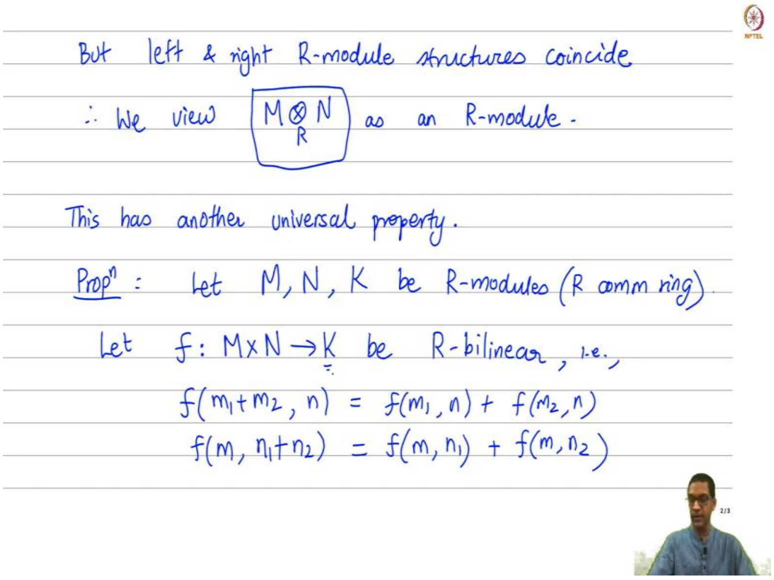

But in addition to these two we also demand that if you put a scalar R, then you should be able to pull the R out, so observe the on the right hand side what do we mean by pulling the R out, the right hand side f of m n, recall is an element of K here, it is going to the module K and K is in R module, so there I know what scalar multiplication means. So, f of r mn is R time's f of m n and this should also be the same as f of m comma rn.

So, now I want this to be true for all m, I also had m1 m2 on the previous page for all n n1 n2 in N and for all scalars r from the ring R. So, any map which satisfies these three properties is said to be, so I need three properties 1, 2 and 3 such a map is said to be R bilinear. And observe this is the usual notion if you have looked at things like vector spaces and so on, which corresponds to the case when R is a field, then this is exactly what we call by linear maps between vector spaces.

So, given such an f, let f be R bilinear, then what is the proposition say, it says there exists a unique R linear map, let us call it f tilde from M tensor N over R to K such that the diagram commutes, so what diagram commute? Such that f tilde composition alpha is f and where what is alpha? Where, alpha is the standard map from M tensor N to M cross N M, sorry, M cross N to M tensor N. So, this is the usual map.

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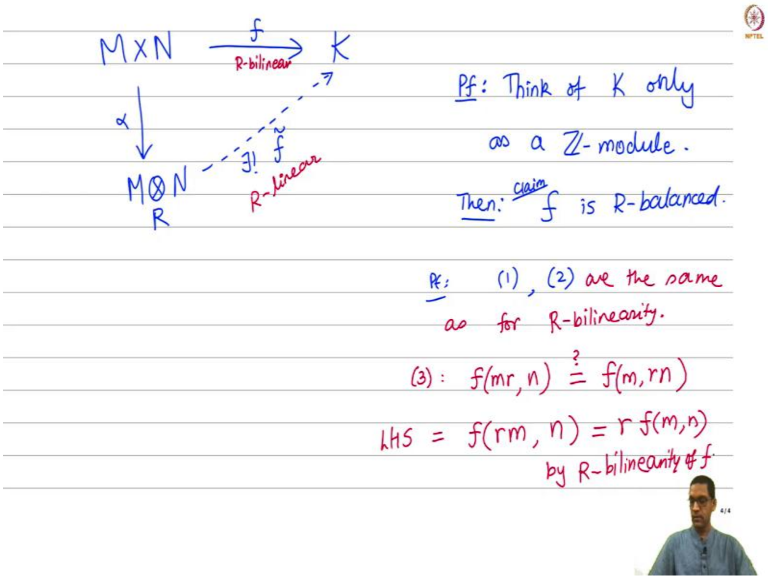

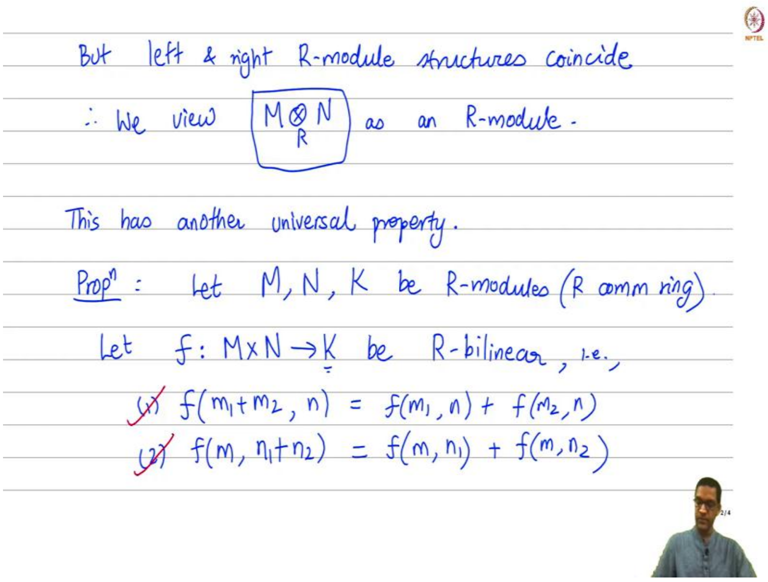

In pictorial terms this says that you take M cross N to K with given a bilinear map f, then look at the standard says that this map f gives rise to a unique map f tilde, the key point here is that this is R bilinear, whereas this map f tilde that we have defined is an R linear homomorphism between these two R modules. So, let us prove this. In some sense, the existence of the map itself comes from the usual property of the tensor product, so let us first look at that.

So, observe that an R bilinear map is automatically R balanced, so first, I will reduce this to something before, so observe now that think of this K here think of K only as a so think of K only as a Z module for now, forget the fact that it is an R module, for now temporarily, then I claim that the map f being R bilinear in fact implies that it is an R balanced map, claim f is actually R balanced, why?

Well, the R balance-ness the property is 1 and 2 that you need for R balance-ness are exactly the bi-linearity conditions 1 and 2 that are satisfied. So, these are exactly the two conditions you need for something to be R you know the first two conditions you need, so 1 and 2 are the same as for R bi-linearity, it is only condition 3 that is different for R balanced maps, so what is condition 3?

We need that f of mr comma n has to be the same as, so the question, is this true? Is this is the same as f of m comma rn? Where r is in R and m and n are from M and N. So, why is this why does this follow from R bi-linearity? So, let us check. So, what is the left hand side? Well, f of mr comma n, so what is mr mean? I am going to do write module action by R, but recall the right module action is just the same as the left module action. And now by bi-linearity I can pull the R out, so the last step is because f is bilinear, R bi-linearity of f.

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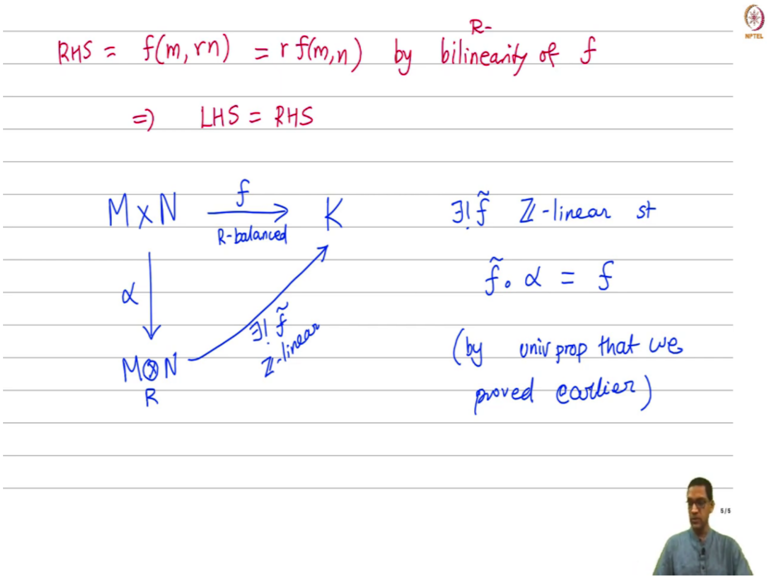

Now, let us compute the right hand side, similarly the right hand side is f of m comma rn, but then by bi-linearity I know that I can pull the r out, this is the R bi-linearity of f. So, what this means is of course that LHS is equal to RHS. So, in fact if a map is R bilinear, it is definitely R balanced. So, what does it imply? It says, now that we can use the universal property of the tensor product that we already proved, what is that?

So on the one hand M cross N to K where K I am going to only imagine is as Z module, I have this map f that is given to me, I know it is R bilinear but in particular that means it is R balanced. So, I have the map alpha M tensor N over R and I know by the there exists a unique f tilde and what sort of map is this I know the usual universal property says f tilde is a Z linear map from M tensor N to K and f tilde makes this diagram commute.

So, there exists a unique f tilde Z linear as in the diagram such that f tilde composition alpha is f. Why is this? By the usual universal property, by the universal property that we already proved. And remember this universal property is actually even under the assumption that the ring is non-commutative, generally for non-commutative rings and for balanced maps you can always find such a linear map f tilde which makes this diagram commute. But now we are saying if further the ring is commutative and the map is not just balanced but also bilinear then f tilde acquire some additional properties and the additional property is that it is R linear, so that is the only additional thing we need to check.

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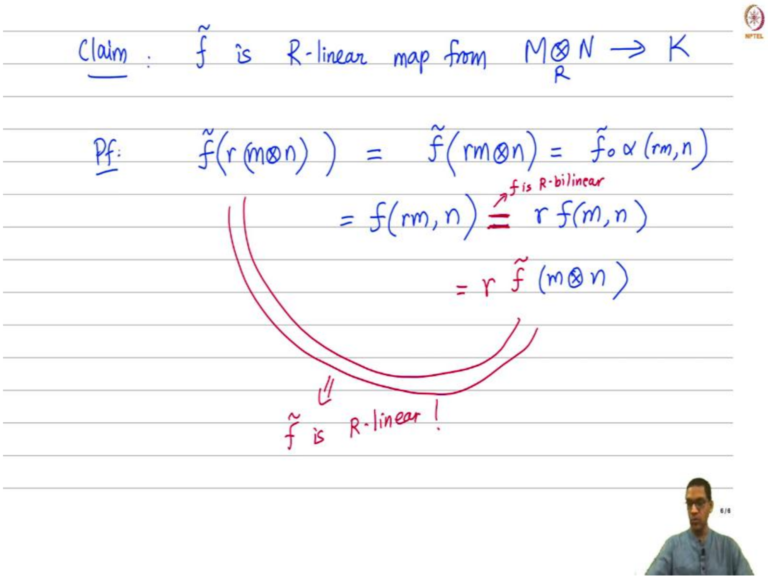

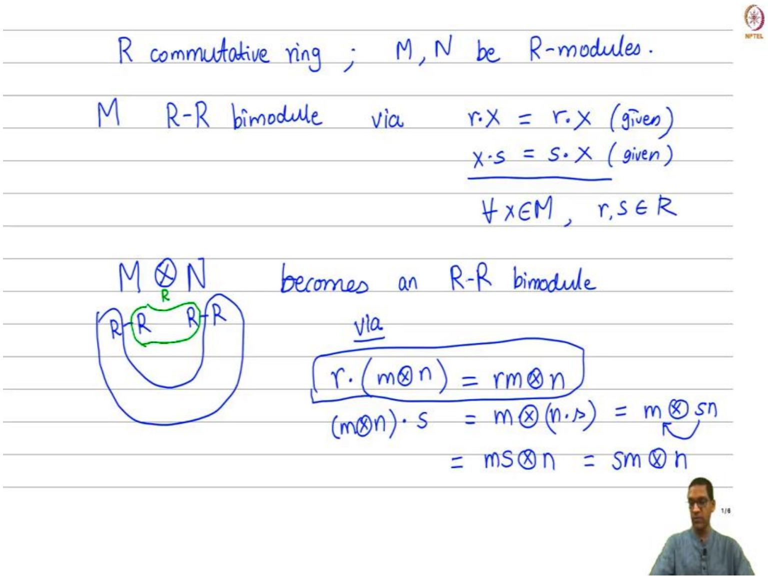

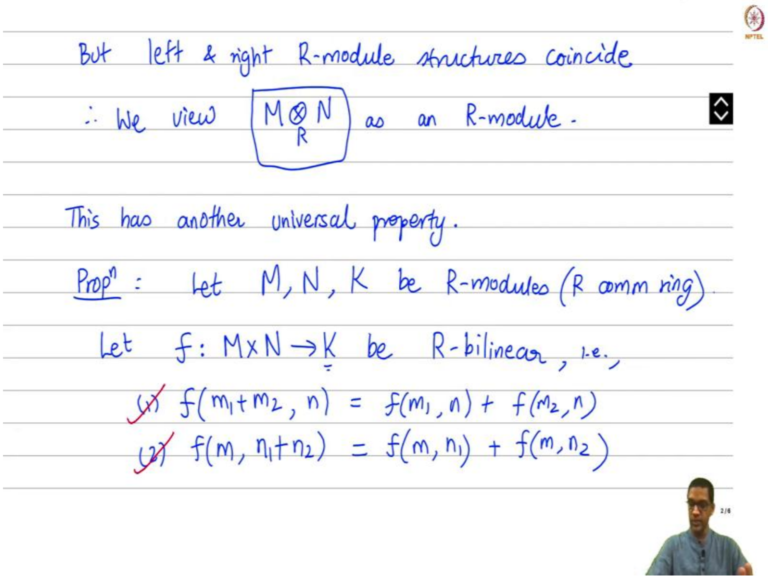

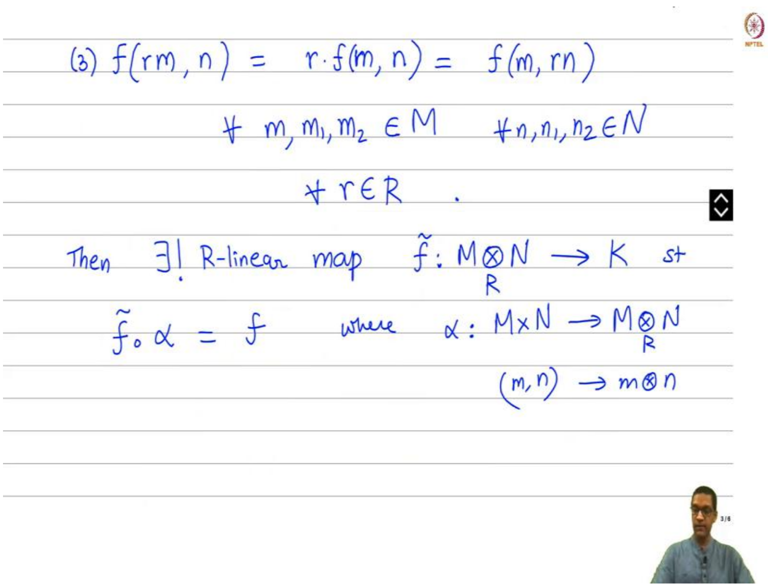

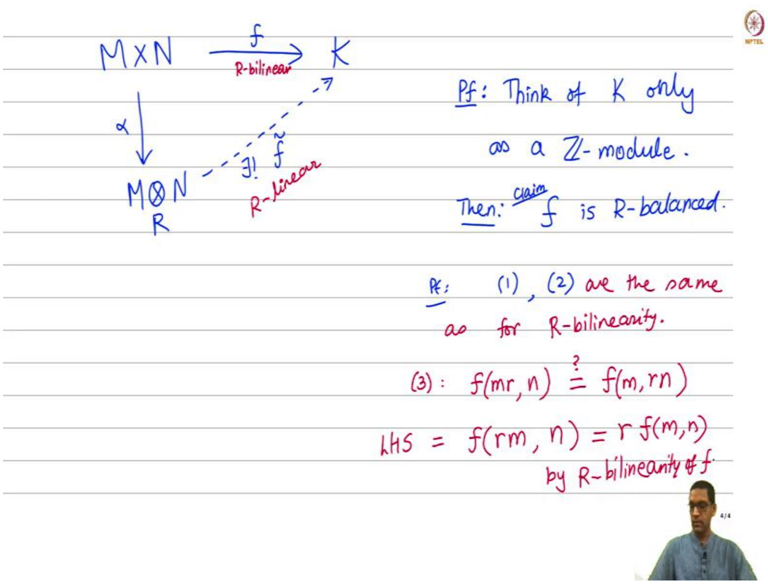

So, claim f tilde is not just Z linear it is actually an R linear map between these two R modules, it is an R linear map from M tensor N to K. So, let us prove this. Now, how do you check R linearity? Let us just check it on the generators. So, I will take f tilde, let me take a typical generator m tensor N and I will hit it by R and see what I can get out of them. So, well this by definition is f tilde of so how do you take m tensor n and multiply it r on the left, what is the R module structure?

So, this is what we looked at the beginning, how do get an R module structure on here it is, an R module such as you can multiply say the first component by R or you can also multiply the second components, all the same, it gives the same answer. So, let us use that, so this is just rm tensor n, well but what is that? That is just f tilde composition alpha of rm comma n.

So, since it makes the diagram commute this is just the same as f of rm comma n and this is R bilinear so I can also pull this out, so it is r times f of m comma n. So, what this means is that f tilde of this, so what is the final answer? This is R times f tilde sorry what did I want to do? So, I just wanted to rewrite this, so f of m comma n is of course, just m tensor that is what I want.

So, this property, so this equation here is where I am using the additional property of f, because f is bilinear. So, now what have you finally managed to prove? I have shown that f tilde sorry I should be a little more careful so f of m comma n is the same, f is the same as f tilde composition alpha, so this is just r times, f tilde alpha of mn, so this sorry there is an f tilde here, it is f tilde of m tensor n.

So, look at this, this final equation here, so this says f tilde of r time something is r times f tilde of that. So, this just tells me that f tilde is not just Z linear, it is actually R linear. So, that is what we set out to prove that this map is R linear. Now, so one can in fact define and that is usually often how it is done when one only cares about commutative rings you can just define the tensor product like this, if you are looking at commutative rings.

So, if R is commutative then you can say that the tensor product m tensor n is the object which has the following universal property that whenever I have an R linear map let us look at this given an R bilinear map from m tensor n to K, it induces a unique R linear map from m tensor n to K. So, this is now you know this is sort of working inside the category of R modules where R is commutative. And this is often how it is defined, but this is very special to the commutative case, you cannot make this definition if R is not commutative.