Algebra – II
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Lecture 66
Tensor Products of Bimodules

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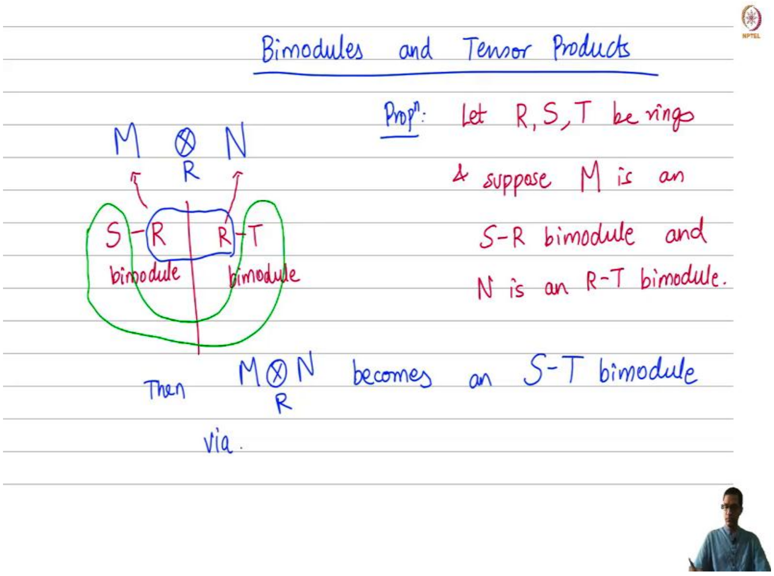

So, last time we looked at what bimodules were and the reason for wanting to look at bimodules is the following proposition, so bimodules and tensor products, so suppose I did the following, so I take the tensor product of M and N over R and in order to do this recall I needed to know that M was an R right R module and I need to know that N, so this is M needs to be a right R module, N on the other hand needed to be a left R module.

But suppose it also happened that in addition to these structures, M also had a bimodule structure, suppose M was not just a right R module but also a left S module such that those two structures commuted with each other, or they had a compatibility. In other words, if M was S tensor R bimodule and similarly suppose N is an R-T, sorry S-R bimodule and N is and R-T bimodule, so what are all these R, S and T, let them all be some arbitrary rings.

And suppose M is an S-T, S-R, so what this essentially means is that in addition to the bare minimum we require in order to construct the tensor product which is the R actions on the right hand on the left, in addition there is some additional structure that both M and N have, then the claim is that then so here is a proposition, proposition with this notation, then M tensor N over R is in fact well it becomes a bimodule over S and T, it is an S-T bimodule, so observe that the residual the S was there on the left and the T was remaining on the right.

So, the sort of the two Rs get together and go away the S and the T remain, observe if you did not have the S and the T, we said in general the best we can say is that the tensor product is a Z module that is all. But if you had this additional action then this proposition says the tensor product becomes an S-T bimodule via, so what is the definition? So, let us say how the actions are defined.

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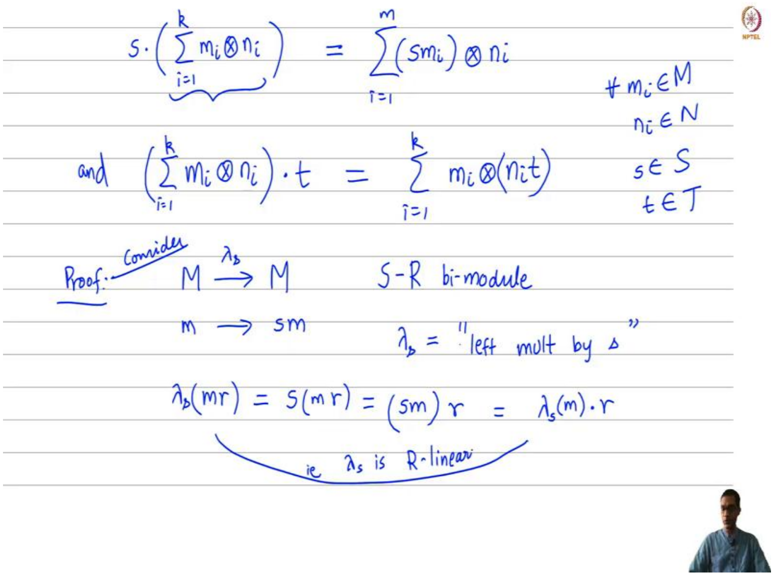

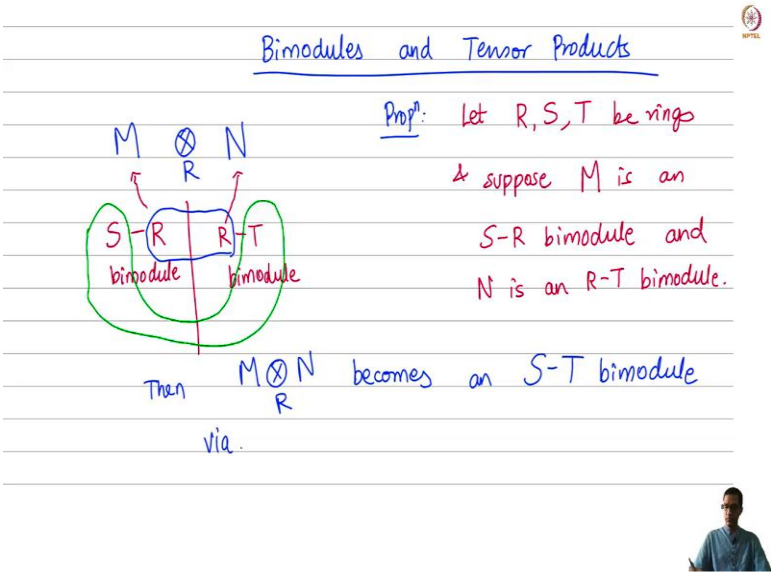

So, suppose I take an arbitrary element of the tensor product mi tensor ni, remember anything can be written like this, though not necessarily uniquely, so suppose he take an element like this, let me tell you what the left s action is, via the following action s acting on this on the left is just you act mi on the left by s, remember m is a left s module this makes sense.

And suddenly the right action satisfies the following property that summation mi tensor ni acted upon by t on the right is just this element and this is for all mi in M, s in S, t in T. So, I claim that there exists an S-T bimodule structure which satisfies this property that on a typical tensor element of the tensor product like this is the left action of s is just given by take the left factor mi in each of those summands, just put an s on its left.

Similarly, for mi tensor ni you put it t on the ni portion, so the claim is such a thing exists, now the point is as we did once before when we talked about functoriality and so, we cannot use these as the definitions per say, the final you need to somehow do something else to show that there exists such a S-T bimodule structure, which satisfies these properties. But you cannot quite start with these as the definition because then we will run into well defined (())(05:22) problems.

Because the same element of the tensor product may have many, many different expressions like this as a sum of mi tensor nis. So, we do not really want to use that, so what we will do like we did earlier is to sort of use the universal properties of the tensor product or rather those the functoriality that we are already proven.

So, let us prove this, let us first define the left and the right multiplication maps so observe firstly that M is a left this is a left S module, I mean it is more than that it is a S-R bimodule, but at the moment let us use only the structure, so what does that mean? It means that for any s in S there exists something called the left multiplication map, so I will call that map as lambda s, so lambda s is just a left multiplication by s map. So, this is a map from m to M.

Now, what kind of map is this? Well, it is certainly is a linear, but in fact it is a bit more observe that if I took m and act on the right by r then this by definition is just S acting on mr, but remember that m is not just a left S module, so now I will use the additional property that I know about m that it is actually a S module and R module and the two modules structures commute.

So, S acting on mr is therefore by the bimodule axiom just the same as sm acting on r, in other words it is just you take lambda s of m and act upon it by r on the right afterwards. Now, what does this tell you? Well, this just means i.e. lambda s is an R linear homomorphism, it is an R linear endomorphism of M if you wish.

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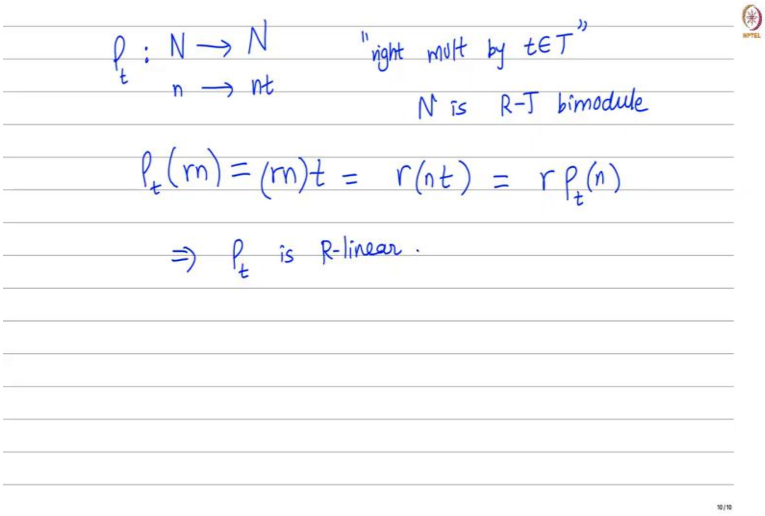

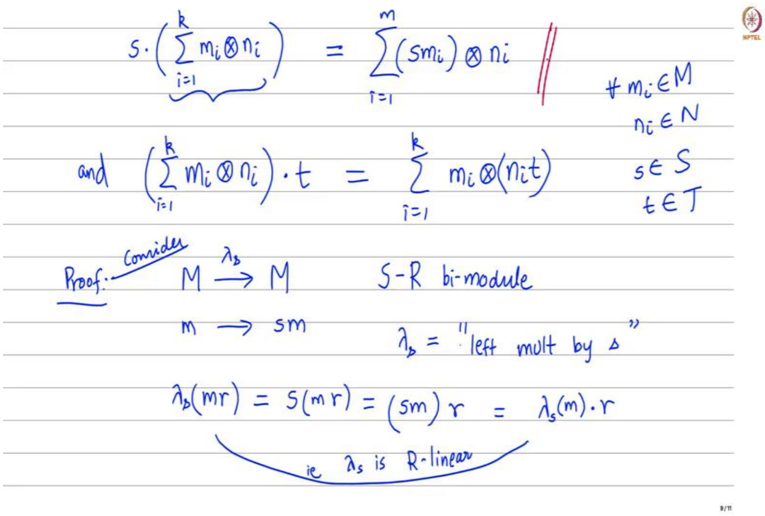

Similarly, we have a right multiplication by T map, so similarly let us also define the right multiplication map, this is from N to N, takes n goes to nt and again this map is also R linear for the same reason, so now N is, so if I multiply on the left by R then by definition this is just rn multiplied by t this is the bimodule axiom it is r of nt, so this is just r right multiplication by t of n. So, this again means that the right multiplication map is an R linear endomorphism of N. Now, the point is when you have these R linear maps, then we can look at the following situation.

So, let me just go back here for a moment, so recall, what is it that we wanted? We wanted to construct, well we wanted to show that there is a left multiplication map on the tensor product which basically only acts on the first component. So, what is this map? Well, it is sort of clear what this map is.

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$$\left\{ \begin{array}{l} M \xrightarrow{\lambda_s} M \\ N \xrightarrow[id_N]{} N \end{array} \right\} \rightsquigarrow \exists\ \lambda_s \otimes id_N : M \otimes_R N \to M \otimes_R N$$

both R-linear

$\mathbb{Z}$-linear

$$(\lambda_s \otimes id_N)(m \otimes n) = \lambda_s(m) \otimes n = sm \otimes n$$

$$\forall\, m \in M,\ n \in N$$

$$\Rightarrow \boxed{(\lambda_s \otimes id_N)}\left(\sum_{i=1}^{k} m_i \otimes n_i\right) = \sum_{i=1}^{k} sm_i \otimes n_i$$

So, consider from M to M I have the left multiplication map from M to M, I have the left multiplication by S map, from N to N let me take the identity map, look at this pair of maps. Now, we have already shown when we talked about functoriality or how there are associated homomorphisms on the tensor product, observe that given such map, so these are both R linear maps, the identity is always R linear.

This means that there is a map called lambda s tensor identity, this is a map from M tensor N to and this is Z linear map, so there exists a map like this, this is Z linear. And when I say there exists a map like this, what is it do? It only acts on the, you know we sort of know what the action is, so recall the definition of this map is the following when it acts on a generating tensor identity on n, this is for all m in M.

And left multiplication by m, by s is of course just sm tensor, so there is such a map definitely which acts like this and that is well with that is how it acts on the generators, so if you see what is this map lambda s tensor identity on summation mi tensor ni then well that is just a summation of the corresponding actions. So, this is exactly the left multiplication action we wanted or we claimed was there, so what we are saying is the left multiplication by s is just given by lambda s tensor t identity.

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Similarly: $id_M \otimes \rho_t$ is a well-defined $\mathbb{Z}$-linear map $M \otimes_R N \hookrightarrow M \otimes_R N$

$$(id_M \otimes \rho_t)\left(\sum_{i=1}^{k} m_i \otimes n_i\right) = \sum_{i=1}^{k} m_i \otimes (n_i t)$$

Need to show: All axioms of S-T bimodule are satisfied.

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Now, similarly analogous to the this analysis just look at identity tensor right multiplication by t, what is this? This is a map, this is Z linear map from M tensor N to itself to M tensor N, what does that map do? It does exactly what we wanted to be our definition of right multiplication which is it only acts on the second component.

So, we have these two Z linear maps, now what you have to show is that so these maps are well-defined at least that is the key point this is a well-defined map we do not need to worry about showing that this definition is well-defined and so on. So, there are well-defined maps all we have to show is that they satisfy all the axioms required to make M tensor N into an S-T bimodule. So, need to show all axioms of S-T bimodule are satisfied. So, of course there are several axioms to check some sense, but you know let me leave some of them for you as an exercise.

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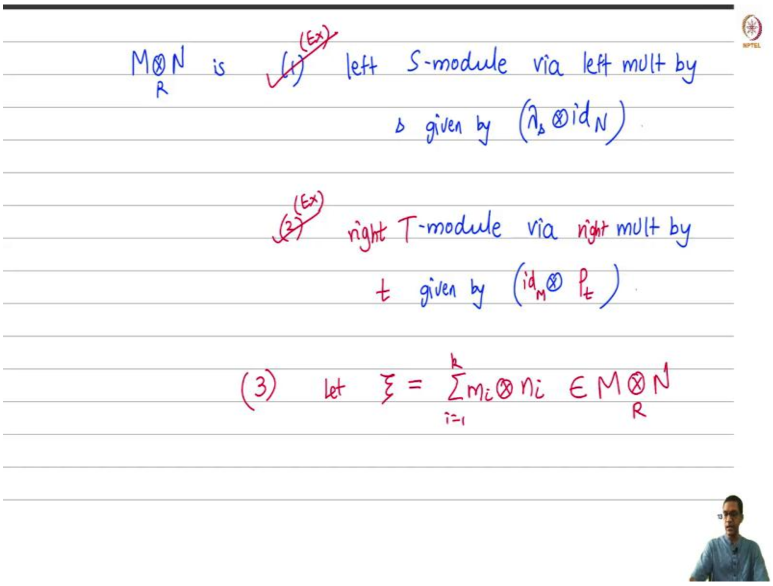

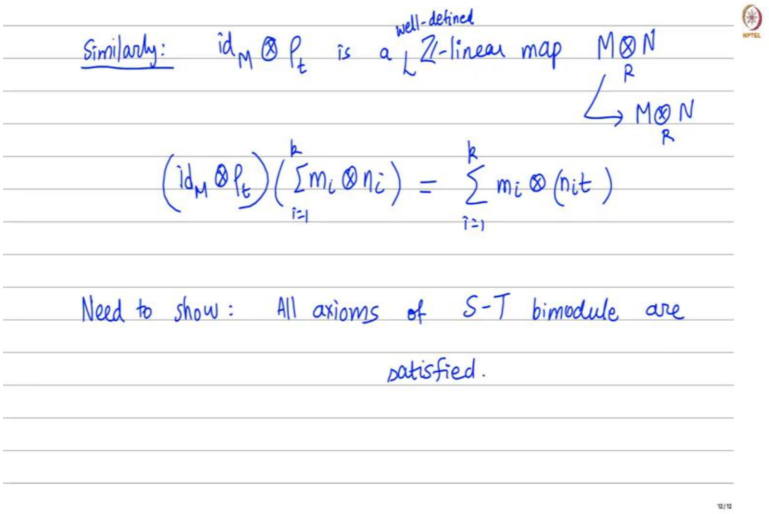

So, first we need to check that this is an S left S module, so M tensor N is let us see what all do you need to check number 1 it is a left S module under the given action, via the action that we talked about, so via are this, via left multiplication given by this, via left multiplication by s given by. Similarly, so this I am going to leave as an exercise. So, let us check it is easy exercise.

Now, similarly the right module structure is also analogous, so that is the other second exercise, check that this is exercise two maybe, which is to check that this becomes a right T module via the right multiplication by an element T given by the other map which is identity tensor right multiplication by T. So, you define left and right multiplication by elements of S and T using these maps.

And then you have to show that the left S module axioms on the right T module axioms are satisfied. So, that is more or less easy, because they are just satisfied on each individual component separately, but the key thing that we need to check is the compatibility between these two. And what does compatibility mean? It says, if I take an element Xi summation mi tensor ni in the tensor product and then I act first on the left by S, then on the right by T, well I get the same answer if I do it the in the other order.

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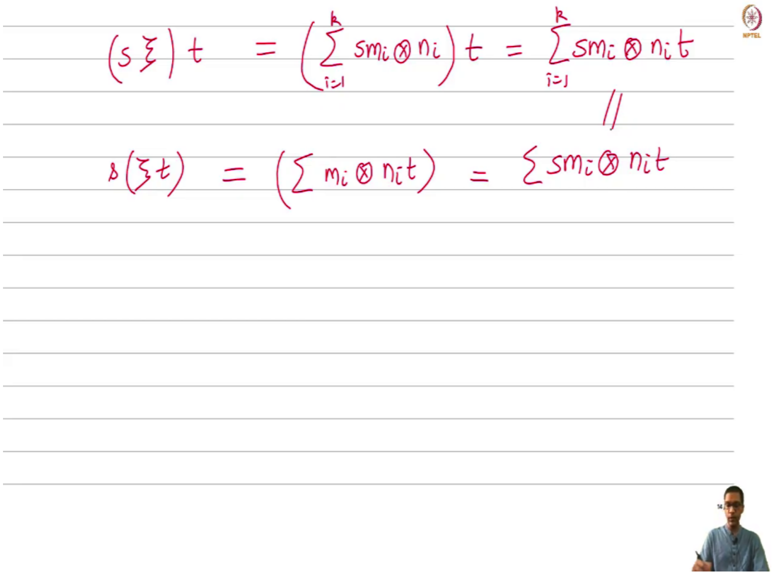

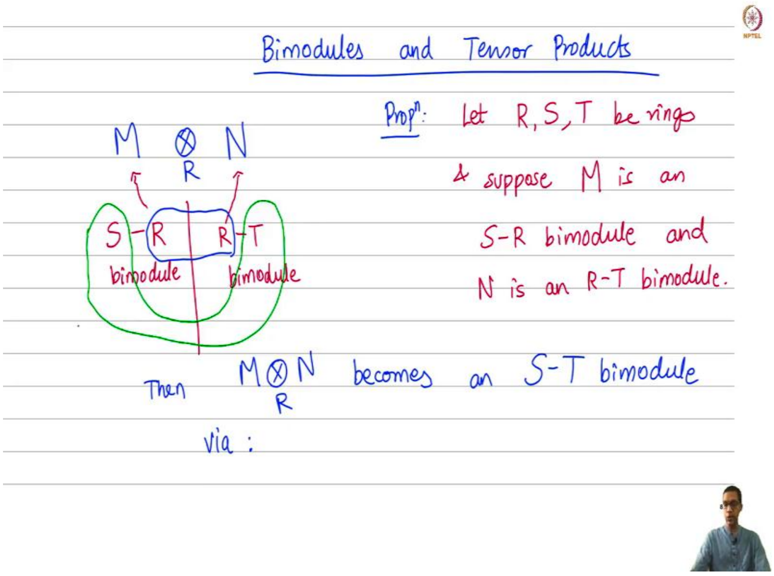

So, let us take this elements Xi act first by s and by t, let us check on the one hand this is nothing but summation smi tensor ni acted on the right by t, but t only hits the second component, so this is just nothing but smi tensor ni t. And observe in some sense what is happened is that the s and the t have acted upon two different parts, the two different components tensor components, so if you did the other order you will still get the same answer, it will just the first step will give you ni t, the second Step will give you a smi that is all the final answer is the same.

So, therefore this is the third and important axiom, which says that it is in fact a bimodule. So, that proves this proposition, so this is a very important proposition in some sense it says that if s and t had more structure, sorry if M and N had more structured to begin with, they were s and t bimodules, then the resulting tensor product will also have more structure. Now, in particular, so let me just quickly mention some of the particular cases.

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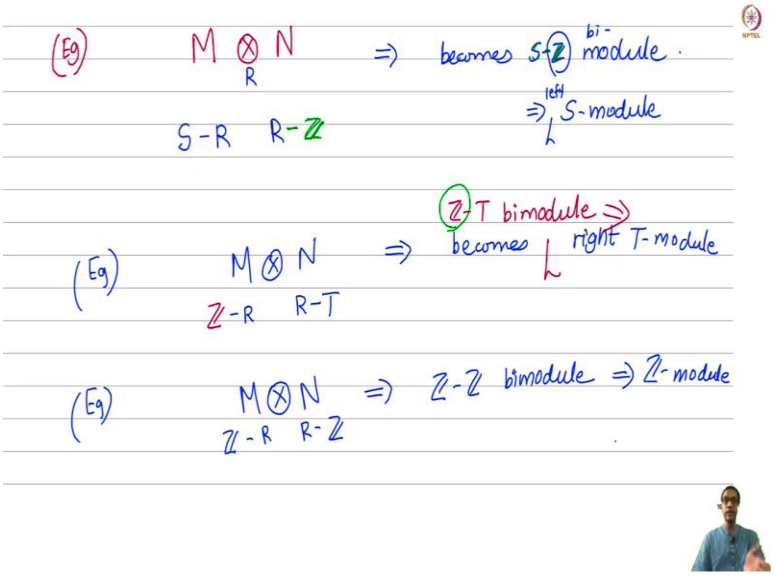

If you only had one or the other, so there are many variations to this theme, so recall, so I suppose I am looking at M tensor N over R, so if M is an R module right R module there is left R module maybe only one of them is has an additional structure, you know M can be an S-R bimodule and M could and N could only be an R module.

Now, if that were the case, then the resulting object, the tensor product in this case, the tensor product becomes well there is only one structure that is left over, it is just an S module, because N did not have any further additional structure it was only a left R module. Of course, the best way to think about this is really in a unified fashion, so here is may be a good explanation for why this happens.

So, think of it nevertheless as a bimodule, recall we have said this, if I have a left module R I can think of it as a bimodule by just thinking of the right action by Z. So, this is an S-R module that is an R-Z module, therefore by what we have already said the final answer, the tensor product becomes an S-Z module, well in S-Z bimodule.

But again having as a Z as one or the other component means you know does not add anything extra to the structure, you already knew that your module the underlying thing was an abelian group it was a Z module, that is the same Z module structure you are talking about. So, the final answer is an S-Z bimodule, but the Z does not add any additional information, so that is just the same as S module.

Similarly, let us just look at the other variation, so if I had M tensor N and maybe M was right R module, it was a right R module, N was say some R-T bimodule then this means that the final end product M tensor N becomes only right T module if you wish. Here I when I say S module I mean left S module.

Again, why is this the case if M did have additional structure, I can always put a Z there, I can think of it as a Z-R bimodule. And therefore the final end product becomes a Z-T bimodule, so the final end product becomes Z-T bimodule, but a Z-T bimodule is just the same as a right T module because this Z here does not add any new information, we already knew this was you know it is the same Z module structure.

So, these are all various ways of remembering this and in particular this goes back to the original structure we had of tensor product if M and N did not have any further structure, other than this was right R module, this was a left R module then I can just put Zs on both sides, thing this as a Z-R bimodule that has an R-Z bimodule, so the end product is only a Z-Z bimodule.

And again the Z and puts no further structure here it turns out just to be a Z module and that was the starting point again. So, these are you know this collection of facts about bimodules is very very important, because usually it is not just the Z module structure we are interested in, M and N will come with often with some additional structure. So, we need to know how that is inherited by the tensor product.