Algebra – II
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Lecture 65
Bimodules

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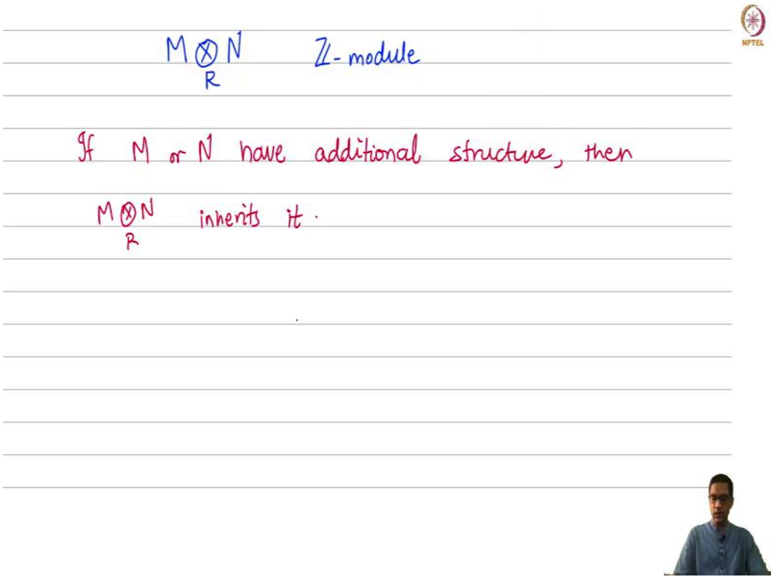

So, we constructed the tensor product and we looked at the functoriality of the tensor product. Now, there is something that we can usually say about tensor products that is important, so let me recall that if M was a right R module and N was a left R module, we define the tensor product and the resulting object was only as Z module in general. But it turns out that the tensor product often has additional structure more than just being an abelian group, there is often more we can say about it.

And that usually comes when M and N have some additional structure to that. So, here is the point if M or N or both have additional structure then the tensor product inherits it, then M tensor N inherits it. So, let us make this a little bit more precise, so to make this precise, I will introduce something in maybe in great generality and then we will look at various special cases of this.

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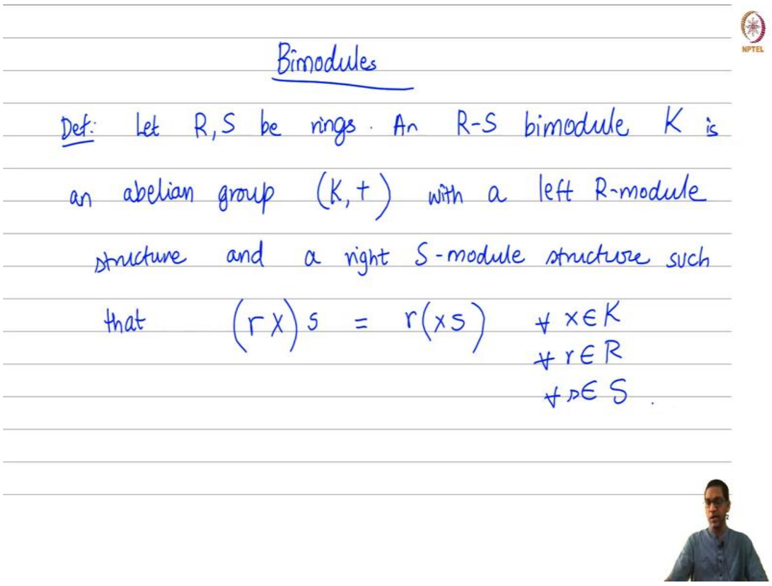

So, this is the notion of bimodules, so we have looked at module so far, which is well it is an abelian group with an action of a ring on it, now a bimodule is the following you need two rings let R and S be rings, an R-S bimodule, so an R-S bimodule K is the following is an abelian group and we will denote this structure.

So, an R-S by module is an abelian group so I am calling the operation as plus with well its got the following structures, it is a left R module, it is a right S module with a left R module structure and right R module structure a right S module structure sorry right S module structure such that, so there is an additional compatibility condition we require between these two structures such that the following is true.

So, what does it mean to say there is a left R module and right S module structure, given an element m in your in K what I can do is I can define maybe I will call it x. So, given an element x in K, I can define left multiplication by a scalar from r, so this element makes sense because I have a left R module structure.

But I take this element and then I write multiply it with an element from S, which again I can do because I assume my module K I mean my set K had a right S module structure, so rx right multiplied by s I demand should be the same as first right multiply x by s and then left multiply by r, this should be true for all elements x and K for all scalars r in R and s in S. So, any such thing is called a bimodule and we say its R-S bimodule, because R is the left module structure and S is gives you a right module structure.

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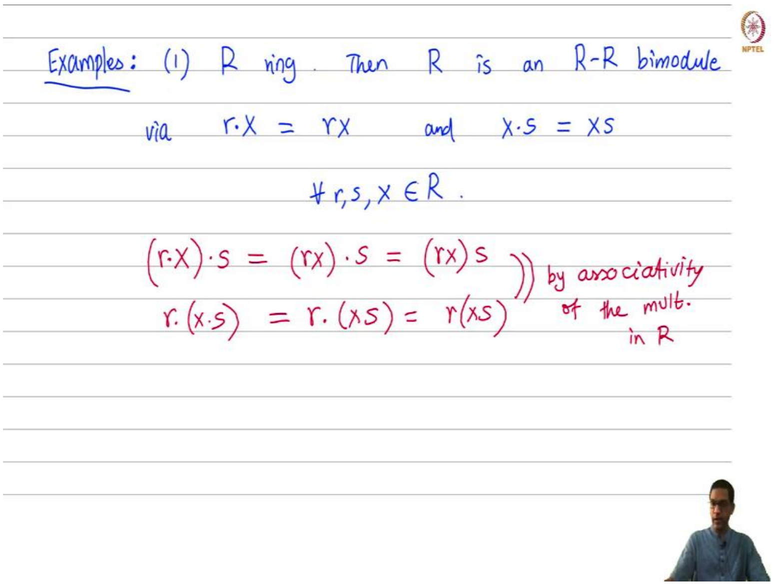

So, here are some examples bimodules by the way are extremely common, they are you know in many, many situations you actually have bimodules, so here is a example if I take a ring R, not necessarily commutative then recall we have already looked at this then so R I claim is an R-R bimodule, then R is an R-R bimodule, meaning it is got a left R module structure, it is got a right R module structure such that the two structures are compatible via what is the definition?

So, given an element x and r, I just define the left action by just multiplication and given x and given an element I will call it s but it is again in r I again define this to be right multiplication by s, so this is for all rs, so claim is that the left multiplication makes R into a left R module as you have seen before, right multiplication makes R into right R module, but the new thing to check here is that these two are compatible, so let us check that.

So, I take x, I first act r on the left and then s on the right. Well, what does that give me? It is rx multiplied by s, but then that is just you know sorry rx acted by s is just rx times s, whereas the other order by the same token it is r multiplying xs. But observe these two are both equal because the multiplication is associative by associativity of the multiplication in the ring. So, that is exactly what makes r into a bimodule over itself because the multiplication is associative and therefore the left and the right R module structures are compatible. So, this is the first example, any ring is a bimodule over itself.

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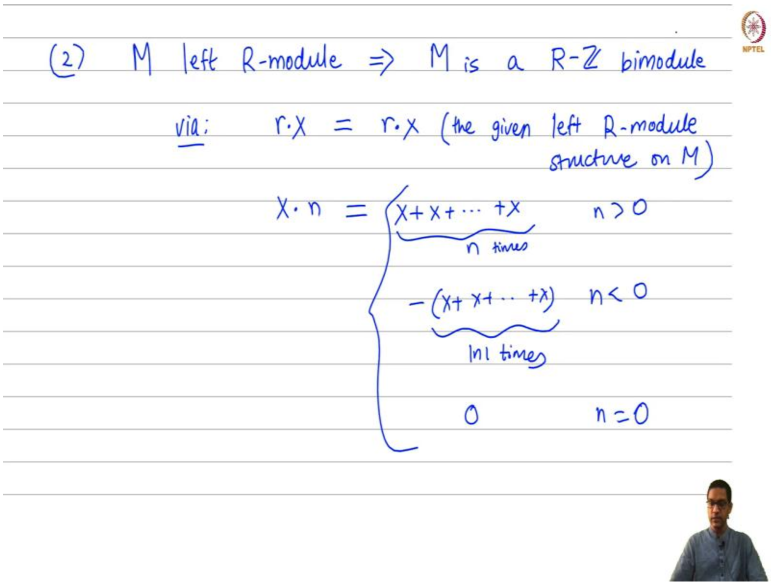

Secondly, so any left module for example, so suppose M is a left R module or similarly right R module you can actually think of it as a bimodule, so a left R module is nothing but bimodule where the ring on the right, I just take to be the integers. So, I claim that if M is a is left R module then I can actually think of it as an R-Z bimodule.

So, firstly what is that require I should be able to think of M as a left R module that is already given, I need a right Z module structure, but remember when I say that M is a module over R, I already know that M is an abelian group. And an abelian group is of course is a Z module. So, here is Z being abelian I do not care whether it is left R module, left Z module or a right Z module it is the same thing.

So, here is the definition, so how do I think of it as R-Z bimodule via the following definitions given x and m r multiplied by x is just the usual given this is the given R module structure. And then how do you do right multiplication? Well, that is because it is you know how do I write multiply by an integer if you wish x times n, well that is just it is there is also a Z module structure already, it is just x plus x plus x n times this is provided n is positive, so our usual definition, if n is negative you just put minus of x plus x plus x so many times, so this is mod n times and if n is 0, then you just put a 0 there. So, this is the usual definition by which we make something to a Z module, so it is the same definition.

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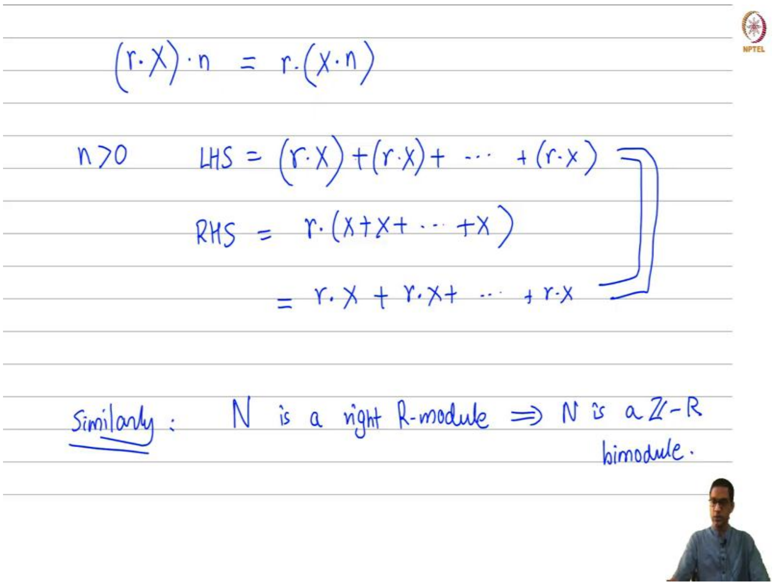

Now, the claim is that these two structures are compatible in other words, if I take x, I first left multiply and then right multiply by an integer, then it is the same as doing this. But then this is just R module property itself, so let us just check this for I will just check it for n positive the others are similar, so what is the left hand side? This is just rx plus rx plus rx n times.

Now, what is the right hand side by definition? It is r acting on x plus x plus x n times. But because of the left module axioms when r acts on a sum of elements, there is distributivity, in other words it just splits into so many terms. And therefore that is the same answer as this. So, in fact this is a R-Z bimodule, so I should also add similarly if I had a right module, so similarly it also has the following if N is right R module, so there is a right action R, then you can still think of it as a N is a Z-R bimodule, can think of it as a bimodule by just making the left ring into just Z. So, same definition here. So, this is the second important example.

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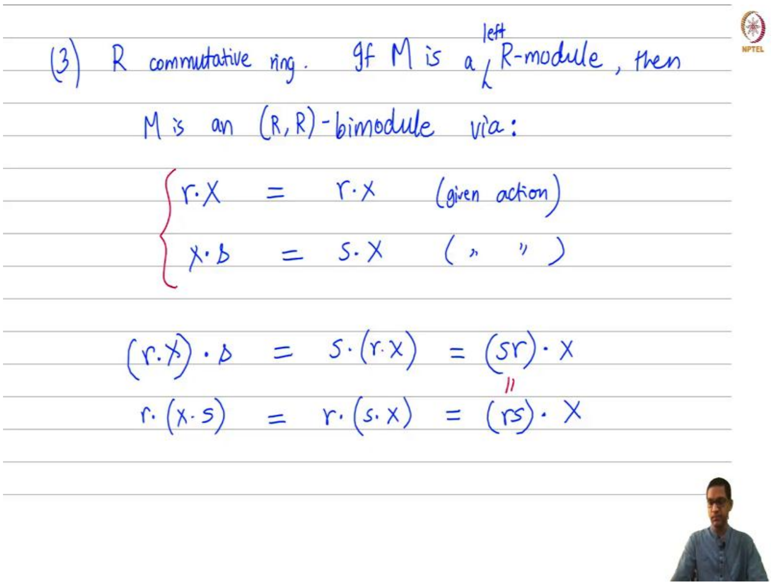

Now, here is the third example, which is that if R is commutative and this is a very frequently occurring and very important example, if R is a commutative ring then an R module a left or right R module, so if M is an R module, recall here left or right is the same we do not need to worry, if M is an R module then in fact you can also think of it as a bimodule then M is an R, R bimodule, how? Well, given an element x of m you must define what left multiplication by R is, this you just define as whatever is given because M is already in R module, so you can think of it as let us say you think of it as a left R module. So, this is the given action.

Now, how do you define right multiplication? Well, you still think of it as the same given action, it is also the given action but now you act on the left in some sense. Now, let us check compatibility, so if I do this rx followed by s, well then by definition this is s acting on rx, which is by the action of by the axiom of the module because it is R module I know this axiom here, similarly if I first act xs then by definition this is r acting on s acting on x, we just use the definition again. But again by the module axiom this is rs acting on x.

Now, the point is r is given to be commutative, so these two things are actually equal to each other sr and rs themselves are equal to each other, so of course the final answers are equal to each other. So, what we have shown is that if you have an R module then in some sense you know this s as you can think of it as a left module or a right module, the given action itself makes it into both the left and right module but in fact the given action also makes it into a bimodule in this way. So, this is again a useful and important special case which will keep occurring in many places, it is useful to think of it like this.

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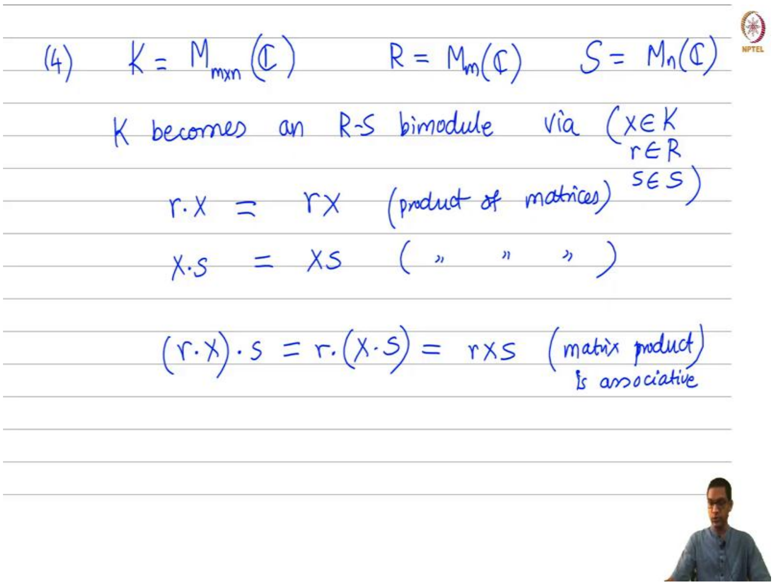

And finally here is the another example which is not any of these cases, so suppose I take the set of all matrices for example, I take the set of all m cross n matrices over some field maybe let us even those are the complex numbers, take all complex m cross n matrices, now you can make this into a bimodule as follows on the left I take all m cross m matrices and on the right I will take all n cross n matrices, then K becomes an R-S bimodule via unless they are following obvious thing I take an element of R sorry of this ring r, so I take an element x from K how do I define a left action? I just take r time x this is now the product of matrices.

Now, this I am defining for all x in K, r in R, s in S. so, I if x is from K that means it is an m cross m matrix and so when I multiply it by sorry a it is an m cross n matrix which I can multiply by an m cross m matrix so it make sense. And the answer is again m cross n, so that is my definition on the left and on the right it is again matrix product but now I take a product in the right that is again well defined and again the answer is still in K.

And observe that for the same reason because you are multiplying on two different sides whether you first left multiply and then right multiplying or do it the other way around, the answer is the same because matrix product is associative, so I can call it this, this is just product of matrices matrix product is associative. So, it is just the product of those three matrices, it does not matter which order I take. So, these are all examples of bimodules.