Algebra – II
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Lecture 64
Functoriality of the Tensor Product

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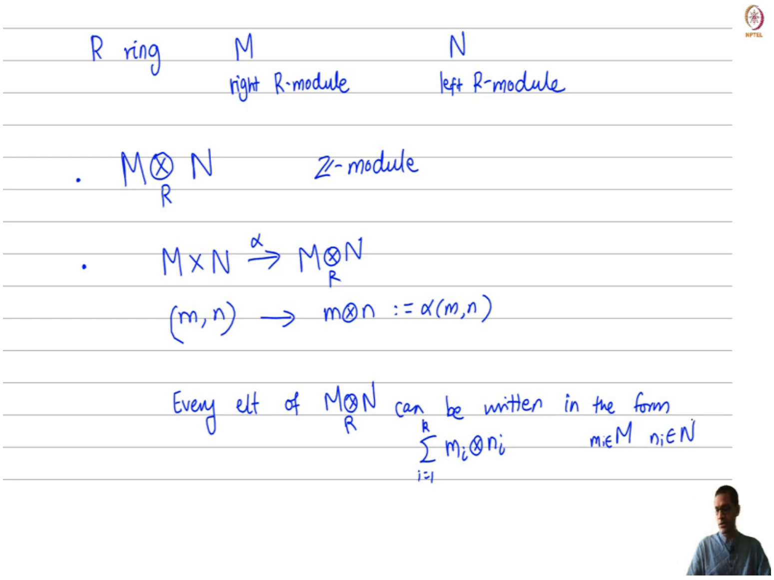

So, last time we constructed the tensor product of two modules, so let us work in the setting of R is a ring and we have right module and a left module, so this is a right module, N is left R module and given these two ingredients we define something called the tensor product of M and N over R and this tensor product finally had the structure of a Z module or abelian group.

Recall also the fact key fact that we used while constructing this tensor product is that there is a an important map from M cross N to the tensor product which we called alpha which is which takes each pair m comma n to well a certain element alpha of m n and the notation for that element was m tensor n. So, this was just notation for the image of m n under that particular map alpha which we constructed.

And the key point about such elements is that well not all elements of m tensor n can be written in this form, the image of alpha is not all of M tensor N, but it is rather close meaning the sort of the span of these over Z will give you everything in M tensor N, so every element so each element, every element of M tensor N can be written in the form summation mi tensor ni, i goes from 1 to k, mi ni, mi come from M, ni come from N.

So, it is a finite sum of such thing this is not a necessarily a unique expression, the same element can be written in many different ways, but what this says is that these elements who form M tensor N they sort of generate the group overs Z, they generate the Abelian group M tensor N. So, often when we want to prove many things, it is enough to sort of prove it on the generators.

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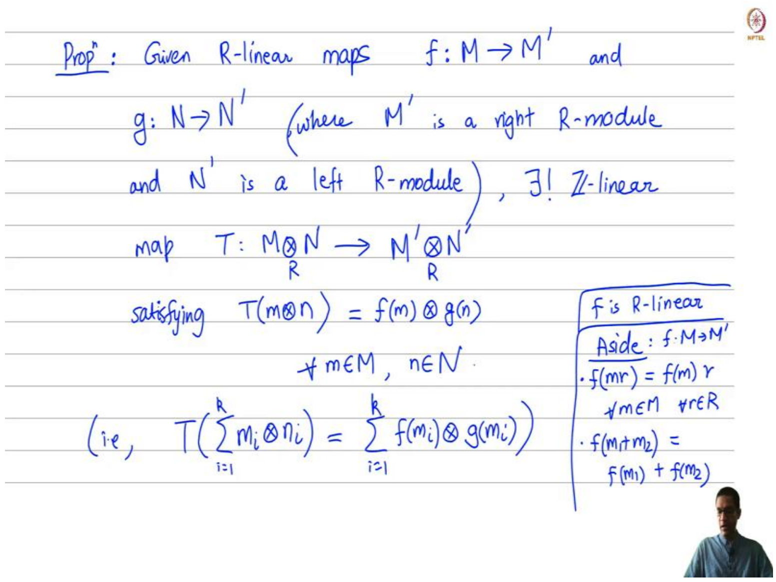

So, we have constructed the Z module M tensor N the goal of today's lecture is to see what happens when you have maps or functions or homomorphisms of M and N. So, suppose I do the following given R linear homomorphisms, given R linear maps, so what am I going to give you a map from M to another right module M dash and g from N to N dash, where what are these M dash is right R module and N dash is left R module.

So, now and what is an R linear map? Well, it is the usual definition also for right modules it just says that if I have f of so aside what is so in this case what is what is meant by right R module homomorphism, it just says if I have f of mr then this is nothing but f of m times r. Now, this should be true for all m into M for all r in R.

So, such a map from M to another right module M dash is said to be an R linear map or a homomorphism. So, an f which satisfies this and the other property the linearity plus f m2 this is true for all m1 and m2 in M. So, such a thing is called an R linear map, so this is R linear. So, same definition as you would give for left modules, except that everything acts on the right.

So, given two R linear maps from M to another right module M dash and from N to another left module N dash there exists a unique well now it is a Z module homomorphism, there exists a unique Z linear map and this map is from let us call it something T for now, it is a map from M tensor N over R to M dash tensor N dash over R, there is a unique map satisfying the following property that on those generators the elements of the form m tensor n, so sometimes such elements are called simple tensors or decomposable tensors, so on a simple tensor m tensor n T takes the following value it is just given by the product well the tensor product f of m tensor g of n.

So, the claim has given two maps f and g you can construct a third map T which has the following action on the generators. Now, the point is to prove a fact like this you cannot just start with this as the definition, so starting you know just suppose we say okay let us define T like this, let us take elements of form M tensor N and define T in this manner, the trouble then is showing well defines, because the M tensor N the decomposable or simple tensors they are they generate the module but a given element of the module M tensor N could be written in many different ways as a some of mi tensor ni.

So, if you define it in this way then you will be stuck with the problem of showing that no matter which way you take of writing that element as a linear combination or as a sum of a mi tensor ni the answer gives you would be the same. So, that sort of messy and we would prefer not to do it in that manner.

So, it satisfies this property for all these guys, so in other words are i.e. on sum like this what we mean is that T has the following property, on such a sum it maps it to this. So, like I said this is the final property that we want T to satisfy, but we cannot start with this as the definition because then we will have to somehow show well defineness. Instead we will do this in directly by using the universal property of the tensor product.

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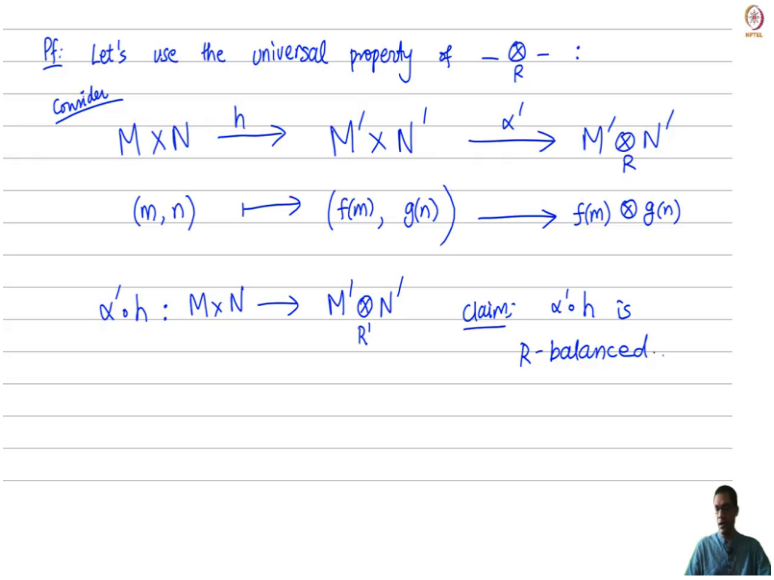

So, proof, let us use the universal property of the tensor product, so tensor n over R of two modules, what was the universal property? Well, it said the following, so if I take M tensor N and I define a map to any other module which is Z by linear then there exists a unique map from the tensor product. So, I mean I should also take these R balanced maps in this case, so let us do the following.

So, let us take M tensor N over R, so I am doing everything over R here, so let me consider the following map, if a M tensor N I can define a map to M prime tensor N prime first, so how am I going to find that map so that we will come to this next. So, how am I going to find this map? I will take a pair m comma n and map it to f of m comma g of n, now from here of course I already have a map to M prime tensor N prime over R, what is that? That is the corresponding map alpha prime for the tensor product.

In other words, this will take this to the element fm tensor gn of N dash. So, let us give this guy a name, so this is a some map h, now consider the composition of these two guys, so consider alpha prime composition h, now where is this a map from? It is a map from M cross N to this module M prime tensor N prime, I claim that this map is R balanced, alpha prime composition h is, it is R balanced. So, we call R balanced meant that its usual Z bi-linear map, but also it has that property with respect to scalars from R. So, let us prove that it is R balanced. So, first thing we need to show is that this is Z by linear, so let us do that.

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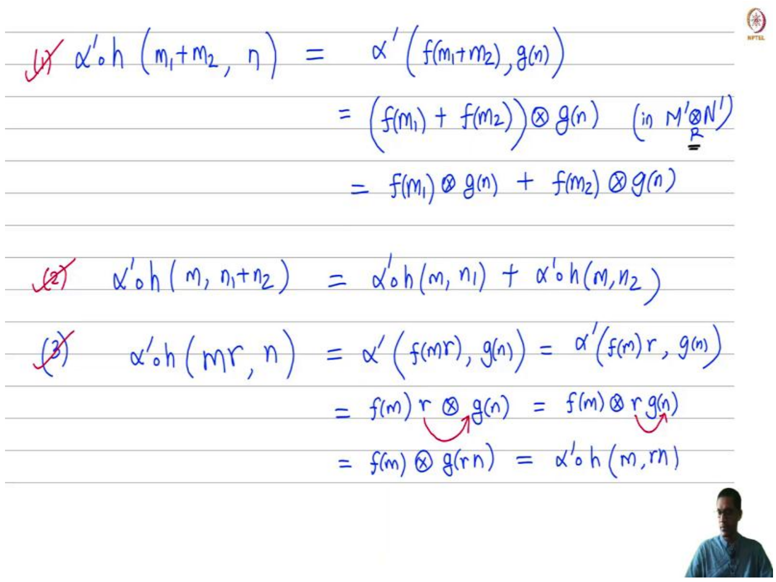

So, let us take alpha prime composition h evaluate it on m1 plus m2 comma n, n should split into two pieces, so let us try this. So, this is alpha prime of f of m1 plus m2 comma n, sorry comma gn by definition of h and this in turn is just because f is given to be R linear, it is going to be the sum so this is now all this is happening inside M prime tensor N prime and there of course this tensor product symbol is bilinear, so we know that this is just going to give me f m1 tensor gn plus, so the first property is satisfied.

So, we have checked that this satisfies the bi-linearity in the first component, the linearity in the first component, now similarly the other one, so let me just leave this as an exercise, this is just equals alpha prime composition h, it is the same proof more or less. Now, the third one the R balanced is what we really care about, now that is the new thing here, so let us check this. So, what are we going to do? We are going to check the following suppose you take alpha prime composition h, you take an element of m, you take an element of n, but let us do the following, let us put an r here.

So, let us write multiply m by an element r from the ring and see what happens to this. Well by definition, this is alpha prime of f of mr comma gn, but f is r linear, recall that just meant that I can put f mr comma gn and again this is f m r tensor gn. But now, recall that this tensor product over R, remove here we are doing all this over ring R that we are looking in that tensor product and so this element here, so alpha prime the map alpha prime itself was R balanced, in other words this R can be moved from here to here that was more or less the defining property there.

So, what that means is that this is fm tensor r gn, but again now g was r linear, so that r can be further pushed inside, this can now go further into the n, so this just becomes fm tensor g of rn, which is exactly what we needed to show, this is alpha prime h of m comma rn. So, what we have used here really is just the fact that you know the tensor product over R is an R balanced map. So, what this means is that the three properties we need are true.

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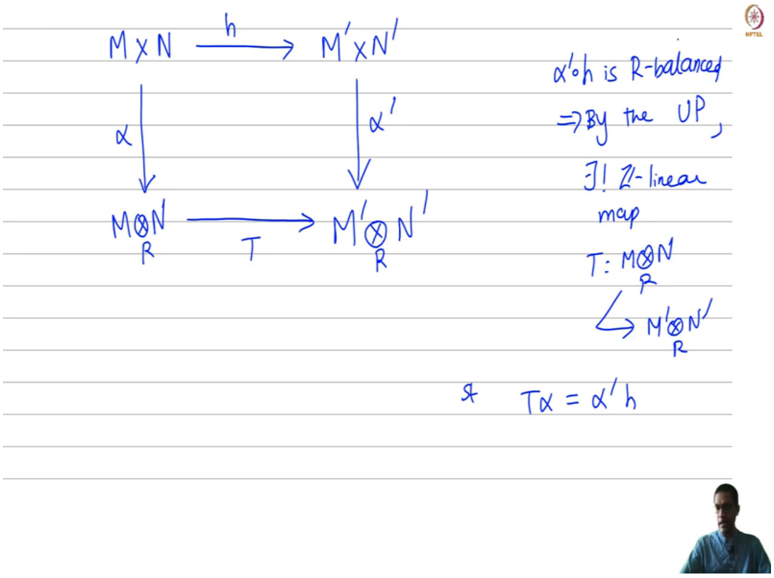

So, we had a map M cross N to so again this is a good way to draw the diagram, so from here have this map so this alpha prime, so this is h and alpha prime composition h was R balanced, so by the universal property, so alpha prime h is R balanced, therefore by the universal property, so I will abbreviate it UP there exists a unique map well this is now a Z linear map, so unique set linear map let us call it T from M tensor N to such that this diagram commute, so there is a map here T such that the diagram commutes. So, in other words T alpha is the same as alpha prime h. So, this just I am using the universal property of the tensor product, but then observe this is exactly what was given in the proposition.

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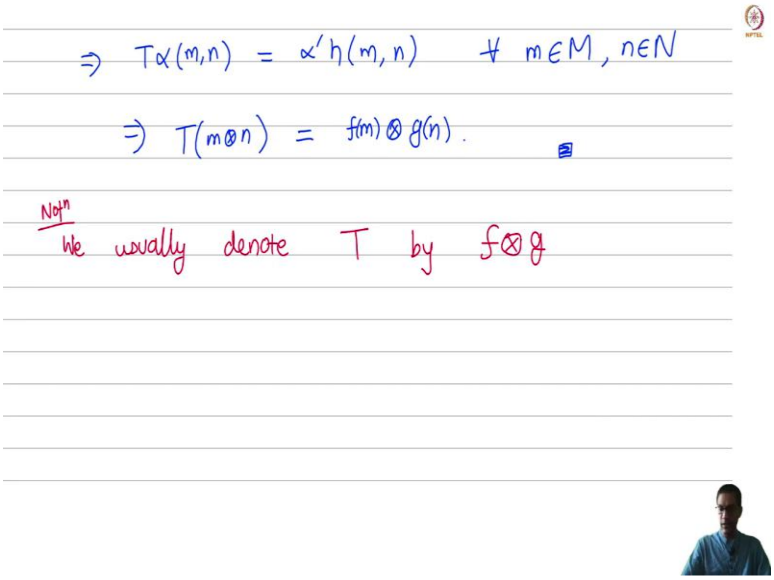

So, T alpha just means if I take T alpha of m comma n is therefore the same as alpha prime h of m n for all m in M, n in N and this is just the same as you know so what is this is T of m tensor n on the left hand side, right hand side was fm tensor gn. So, we have shown the existence of unique map which satisfies this property, in other words which makes the diagram commute. So, that proves the proposition, now let me just say something about notation we usually denote.

So, we so here is a word on notation we usually denote this map T by the symbol f tensor g, so this is sometimes confusing with that tensor symbol and so on, but f and g here are maps, so it is, it does not cause confusion with things of the form M tensor N. So these here are maps, so when I write f tensor g what I mean really is the unique map T which is given by this proposition. So, now there always exists such a map, now why are we doing all this in some sense or where does all this fit?

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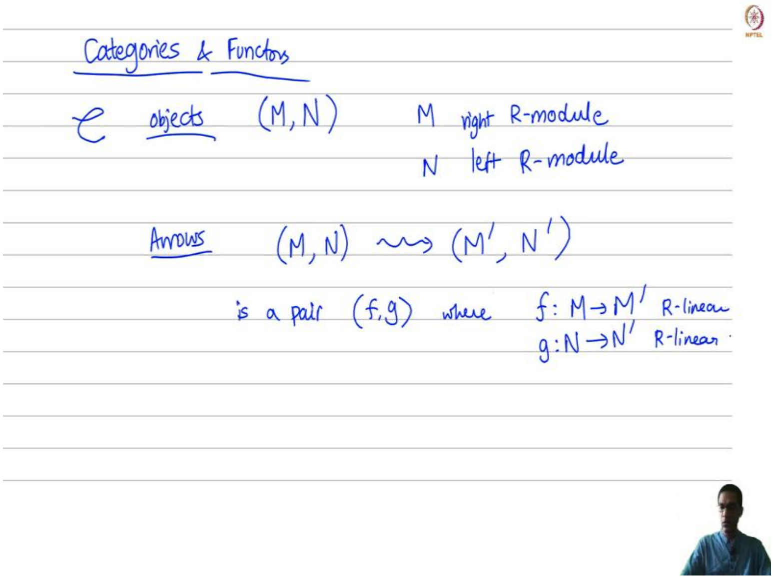

So, here is a remark on categories and functors and where all this fits in that framework, so observe that well in some sense the tensor product can be thought of as follows it is like so okay, so let me define a category C, so the objects of this category are pairs of objects, so what is a typical object in this category? It is a pair M comma N, where M is right R module and N is a left R module, this is like a product category if you wish, you take the right modules as a category, the left modules as a category and what I am forming here is really their product category.

And the arrows now, so these are the objects and the arrows are again you know the arrows as you would have in a product category which is I take a pair M, N and what is an arrow to another object in this category and arrow is just a pair of arrows. So, an arrow is a pair of arrows f, g where, now what is f? f goes from M to M prime, f is an arrow from here, that it is in R linear map and similarly g is an R linear map from N to N prime. So, these are the objects in the arrows in this category.

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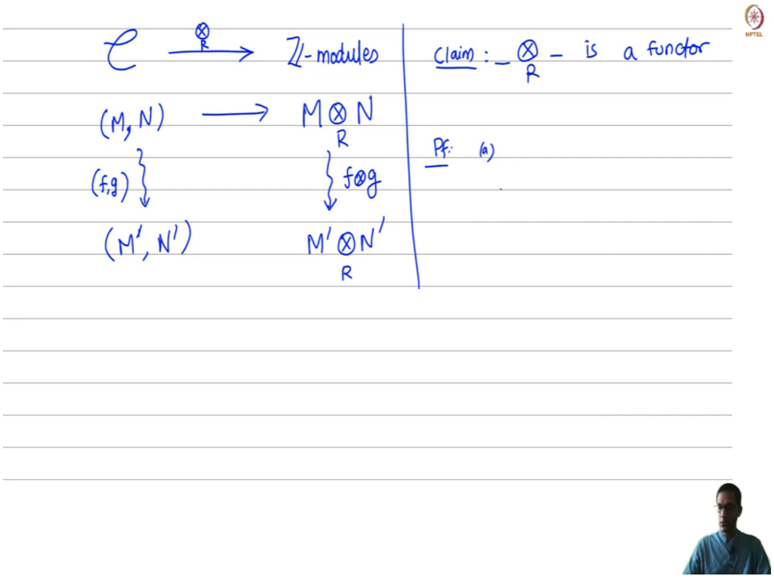

And what we are just saying right now is that the tensor product can be thought of as well it is, at least it is the following it takes objects in this product category C and maps it to objects in the category of Z modules, so here is the, so this is the category on the other side of the Z modules. Now, what is this tensor product map? It takes this pair M, N and maps it to M tensor N over R.

Now, what we have just shown, so this is the tensor over R function tensor over R map what we have just shown is that in some sense it is also a functor meaning while we still have to prove it, but it not only does it map pairs of objects to an object pair, it also maps arrows to arrows, in other words, if I take another pair M prime, N prime in C and I take an arrow between them, but what is an arrow? An arrow is actually a pair of arrows from M to M dash and N to N dash, given such an object, given such a narrow what we have just shown is that there exists an arrow between M tensor N and M prime tensor N prime.

And this is that map which we called f tensor g. So, not only does this tensor product construction map pairs of objects or objects in this product category to Z modules it also maps arrows in the category C to arrows in this category of Z modules. And what is more, this is actually a functor, so claim there is more, this is actually a functor from this category C to this category of Z modules.

Now, recall the definition to show that something is a functor you need to show well more or less two things that it respects the composition of arrows and it respects the identity arrow, in other words, so let us show that it satisfies the two axioms of a functor. So, firstly, we need to show compositions are respected. So, proof a, so let us take compositions, so maybe we will go to the next page.

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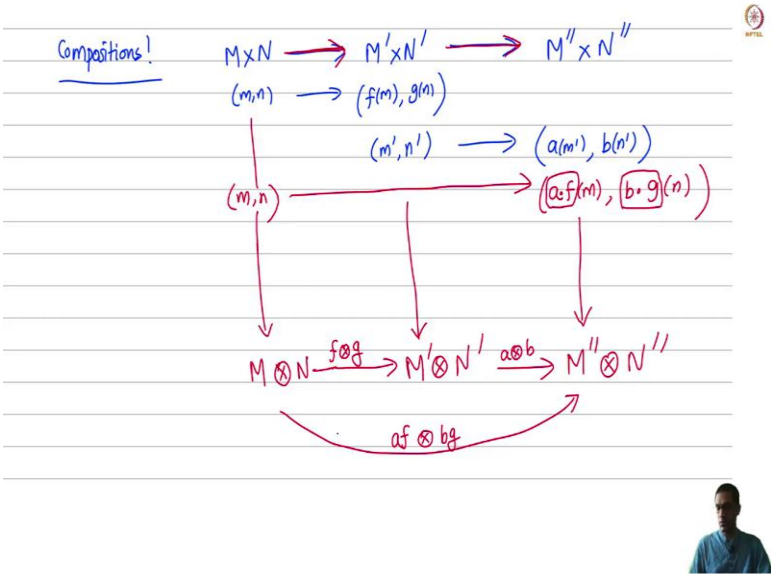

So, first I need to show that this we need to consider compositions, so if I take M cross N let us draw the same diagram as before M prime tensor cross N prime, so if I have I mean this is the arrow, this is the one which takes m, n to fn, gn and suppose I have another one like this M double dash cross N double dash.

And I have another arrow which is it takes m dash cross n dash to let us say a of m dash comma b of n dash, it is again an arrow in the product category, then observe that if you compose the two, so what do you get when you compose these two guys? So, the full composition does the following, it takes m, n and maps it to just the composition of a with f, so it is a of f of m, this is the composition a with f, b of g of n, a and b here are maps. So, it is just the composition of these arrows.

Now, recall how those things worked at the level of the tensor product. So, whenever I am given an arrow, so let us do the vertical ones first, so given this arrow from M cross N to M dash cross N dash, we said well that defines a map between, a similarly there is a map like this, so this is the map f tensor g that is the map a tensor b.

Now, what we need to show to show that this is a functor, the first thing we need to show is that if instead you take the composition of these two arrows in the beginning, which means it you look at a, f and b, g as your new maps from going from M cross N all the way to the other end, M double dash cross N double dash, then the corresponding map that you define here, so what would that be called? That is called a composition f, so which I will write as af tensor bg. So, we need to claim, we will need to show that this af tensor bg map is the same as what you would get if you composed f tensor g and then a tensor b. So, that is the first axiom we need to check. So, let us check that.

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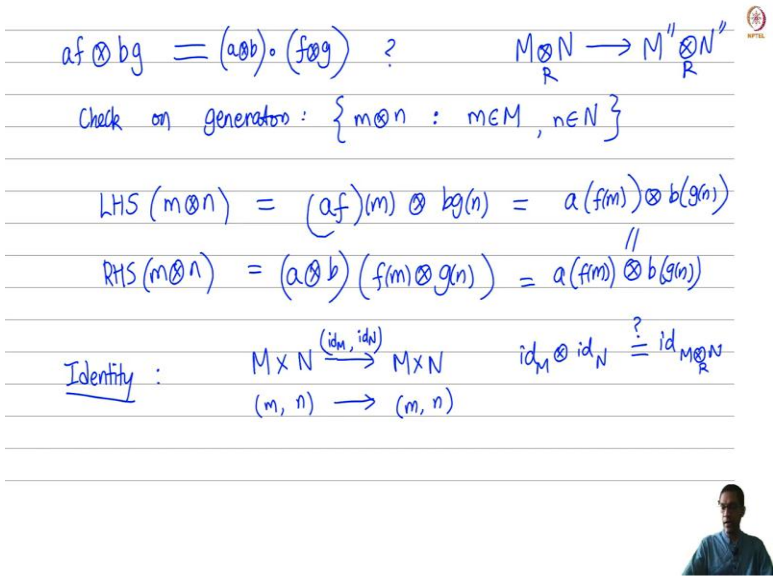

af tensor bg this map is it the same as f tensor g composed with a tensor b. Now, to check a thing like this it is actually enough to check it on the generators, they are both Z linear maps from they are both maps with the same source and the same target, so you just need to show that both maps agree on the generators. So, check on the generators, so what are the generators? Well, we will just take the ones of the form m tensor n, let us just check it on these decomposable tensors.

So, let us apply the left hand side on those guys, if you take the left hand side in you apply it on m tensor, n then by definition this is just af of m tensor bg evaluated on n, which by definition is a acting on f of m tensor b evaluated on g of m. Now, let us apply the right hand side on m tensor n observe this again by definition a tensor b evaluated on again that is a on f m tensor b of gn. So, which is obviously the same, so the left-hand side and the right-hand side are exactly equal.

Similarly, so we have checked the first property for this to be a functor, the second property is that it preserves identities, so that is the second thing we need to check. So, what that means is if I take M cross N from M cross N to M cross N, if I take the identity arrow, so what is the, it should map identity objects I mean identity arrows to identity arrows, so what is the identity arrow in this category C?

Well, that is it is easy to check it is just the identity on M comma the identity on N, it is the pair, it just takes m, n back to itself, this is the identity arrow. So, given this what is the corresponding arrow on the tensor product side in other words what is this? So, let us check what is this, is this just the identity of the element identity is this just the identity arrow on M tensor N, that is the question. So, let us check, so let us evaluate again on the generators that is all we need.

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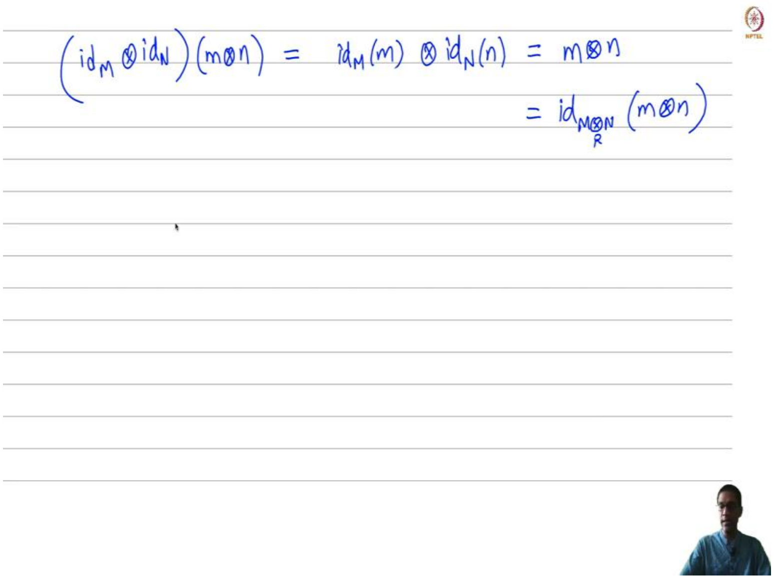

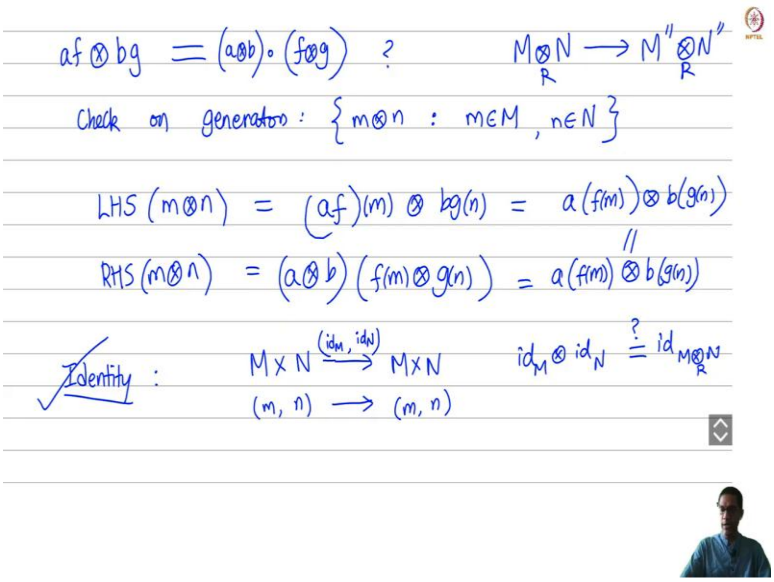

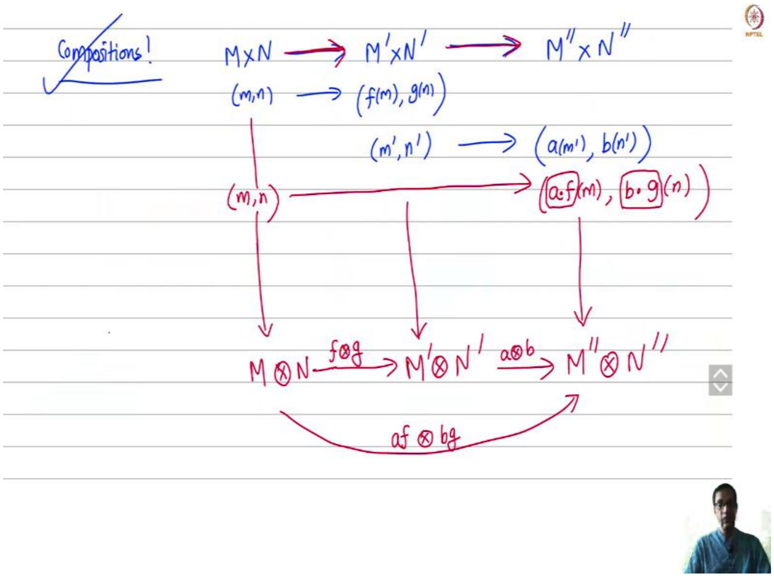

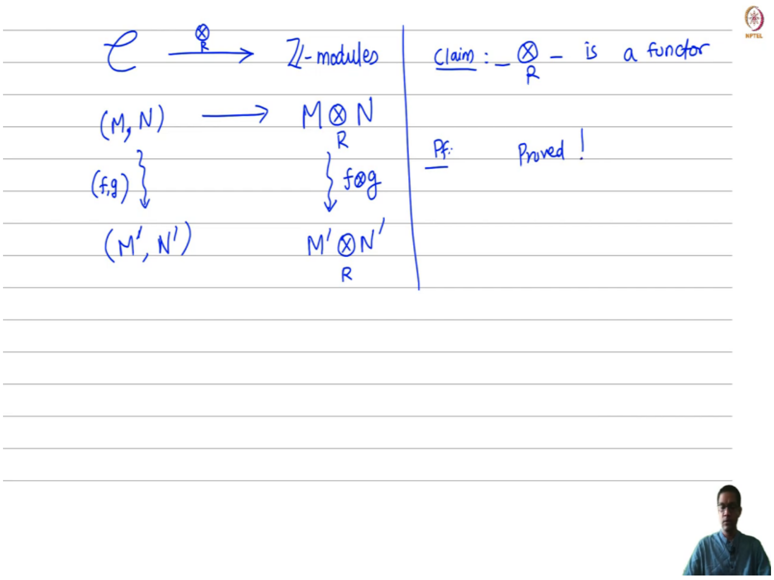

So, identity M tensor identity N, by definition on a generator just gives me, which is just m tensor n, which is therefore just identity map evaluated on m tensor n. So, therefore we have again shown the identity axiom. So, this is also true. So, identity axiom is true. The composition is also respected.

Therefore, what we have is that this is a functor, so proved. So, that is really the point here that the tensor product is actually a functor from the category whose objects are pairs M and N of right and left modules over R from that to the category of Z modules.