Algebra 2
Professor. S Viswanath
Institute of Mathematical Sciences
Lecture 62
Tensor product of R-modules

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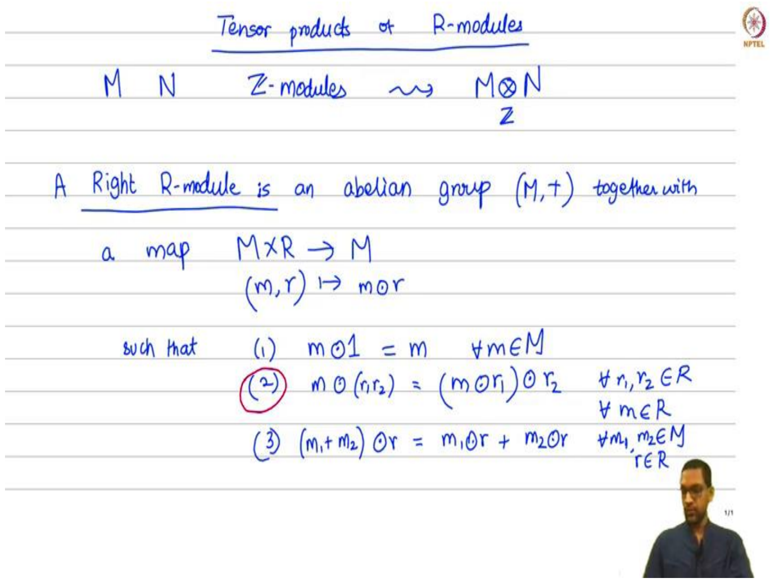

So, today we will start talking about Tensor products of R-modules. Now, until now we talked about the notion of tensor products for two modules, M and N which are both Z-modules. So, given two abelian groups we define their tensor products. Now what we want to do is to sort of do the same thing for modules over other rings now to do this I need to recall briefly the notion of a right module as well because that will enter the definition. So, what is a right R-module?

We know what a left R-module is. Recall a right R-module can be thought of as following. M is a right R-module if it has an abelian group structure. So, an abelian group, so,let us say if this way. A right R-module is an abelian group M, together with right multiplication, together with a map M cross R to M which we will write like this m, r mapping to m may be with a circle on the dot. So, goes to m dot r.

So, this you think of as multiplying on the, satisfying the following axioms such that the following axioms are true m acted upon by 1. On the gives you m, m acted upon by r1, r2 gives you m r1 followed by m r2, followed by r2 and thirdly, if I take m1 plus m2, then that is just the m 2 r and this is for all, m1 m2 in M, r in R. So, the key point here is that this axiom, the second axiom is what differentiates a right module from a left module. If you did the same thing on the left then r1, r2 on m would be, you first act r2 and then r1. So, here you are acting r1 then r2.

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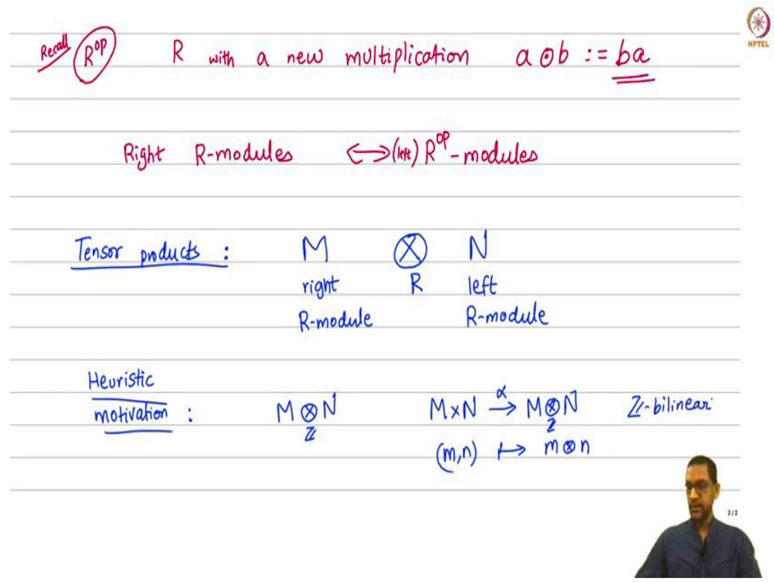

And, so recall that there is also, so there is this thing called the opposite ring. So, recall this from Algebra 1. The opposite ring is nothing but the ring R but with a new multiplication with a multiplication which is redefined as follows. So, you say a multiplying b is now b times a where ba now just means the multiplication in the original ring R.

So, you just reverse the order and multiply and that is a new multiplication and the ring R with this new multiplication is usually denoted as R op. So, the addition is the same. The multiplication is the opposite order and recall the following fact that modules, right R-modules are the same.

So, right modules over the ring R are the same notion as left modules over the opposite ring. So, when I write R-module I always mean left R-modules. This is always left. If I do not specify whether it is left or right then it is left. That is our convention. So, this is a brief introduction or recall of what the R-module structure was.

Now let us define tensor products for R-modules. Now it turns out that to define the tensor product for R modules, what you need actually is a R-module M and a left R-module N. Given this pair what we can define is their tensor product which we will denote the tensor product M tensor N over the ring R. Now just to give you some heuristic motivation for why one would need this sort of combination a right module and a left module in this order, the first component N must be a right module. The second one must be the left module. So, recall, so here is a little bit of heuristics for why we need this.

So, recall that when we talked about tensor products for Z-modules, what we did was, so suppose imagine for the moment M and N are only Z-modules and we defined their tensor product. The thing we did was, there was a bilinear map from M cross N to the tensor product which we denoted m, n going to, so that map was called alpha and we also had an alternate notation. We call this element as m tensor n. Now this map was Z-bilinear and recall Z-bilinear means that if I replace m by m1 plus m 2, it distributes, similarly if I replace n by n1 plus n2, it splits into two pieces.

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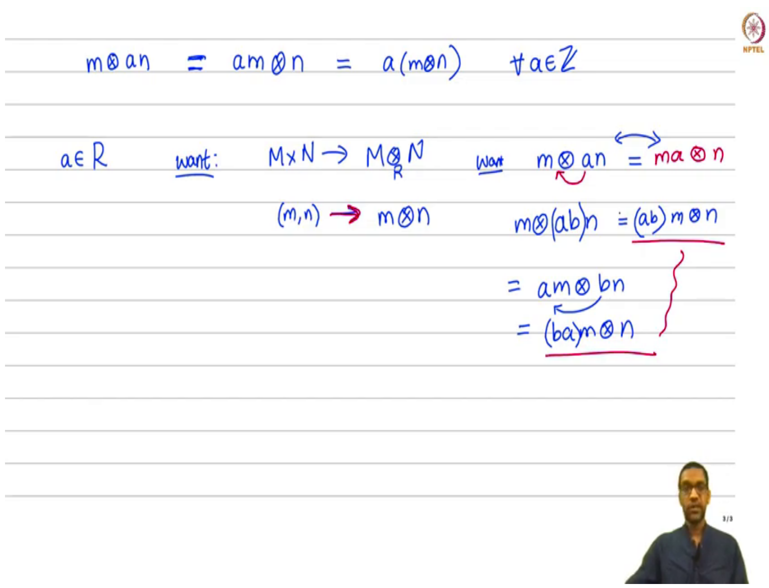

Or another way of saying what bilinearity is, if I multiply n by some integer a, then the scalar a can be pulled out. So, recall that we had this, so multiplication by an integer whether you multiply the first component or the second component it is the same answer and that is the same as this element m tensor n multiplied by this integer a. So, this was bilinearity implied.

But now imagine that you were trying to do this for, you know you would want when we construct the tensor product over an arbitrary ring, we would still want this property to hold. So, here, so suppose for the moment we replaced r, z by r for example. So, imagine now I take the scalar a coming from the ring R. So, yes what I would want my tensor product to satisfy I would want some kind of a bilinear map from M cross N to M tensor N., now what I am going to call M tensor N eventually.

So, this bilinear map I would like it to satisfy, so bilinearity means if I multiply n by a then I should be able to sort of move that scalar out and move it to first component. Like what was true in the case of the integers. So, if this is what I want to happen multiplying n by a is the same as multiplying m by a.

Now the point is suppose I want this to be true but imagine what would happen if you multiplied n by a product of two scalars. Then this wish list, the thing that we want to be true will imply the following. First, I can push the a out, because a b of n is just a acting on b n. So, this will just give me am tensor bn and again I can do the same thing I will get bam tensor n., that is one way of doing it.

On the other hand I can just think of ab as a single scalar. So, m ab n and now by this, what I want to be true that scalar ab can be moved to the fist component. So, on the, so this is what I would get and now observe that if I try, if I want this to be true for an arbitrary ring R, the point is arbitrary rings are not necessarily commutative and so, if my map here, if I want this tensor product map to be bilinear in this sense that if I want to be able move the scalar a from the second component to the first component then I am already in trouble.

So, I noticed that, then this and this must be the same answer. But of course that is not going to be true in general, no reason to expect it to be true because ab and ba are not necessarily the same thin in a non-commutative ring. So, this is sort of the trouble here and this is sort of why you cannot really take the tensor product of two left modules, it does not make sense. This is a remedied. This problem is remedied by making the first component a right module and the second component a left module and in that case this problem does not arise.

So, what we will do is say that, we would like our tensor product to have the following property that I can push this scalar m, a across and multiply it by m but m is only a right module. So, this is what I want to be true. So, I multiply m by a, the scalar multiplication on the right and then you will see that this problem goes away. So, let us go ahead and make the formal definition.

This is just a heuristic motivation for why one module needed to be right and the other needed to be left.

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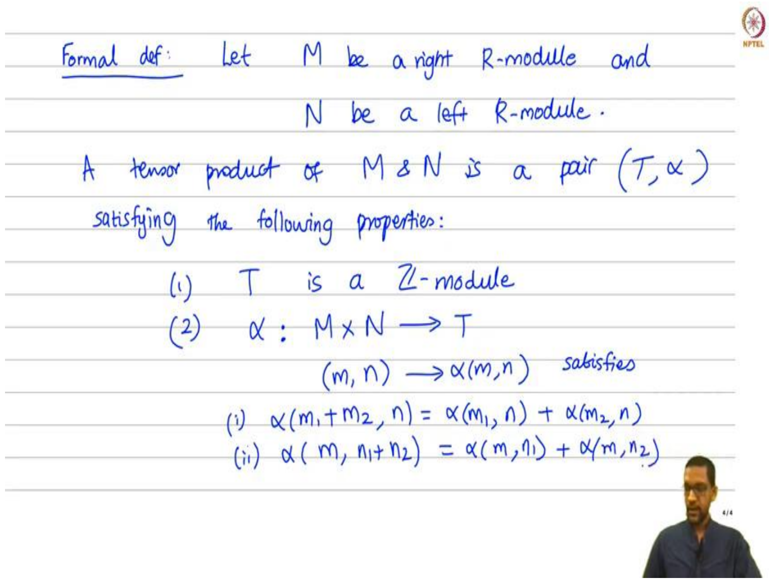

So, here is the formal definition of a tensor product, let M be the right module and N be a left module. A tensor product of M and N is a pair again, like in the case of Z-modules, so what is it? It is a pair P, alpha, maybe we will call it T, alpha, which satisfies a certain universal property. Satisfying the following properties.

One, so what is T? T is a Z-module. So, if we take the tensor product two R-modules the answer is only a Z-module. It is only an abelian group. What is alpha? Well alpha is a map from M cross N to T which has the following property, m, n maps to whatever it is alpha of m, n satisfies, alpha satisfies bilinearity in the following sense that if I replace m by m1 plus m2 then I get a sum, if I replace n similarly.

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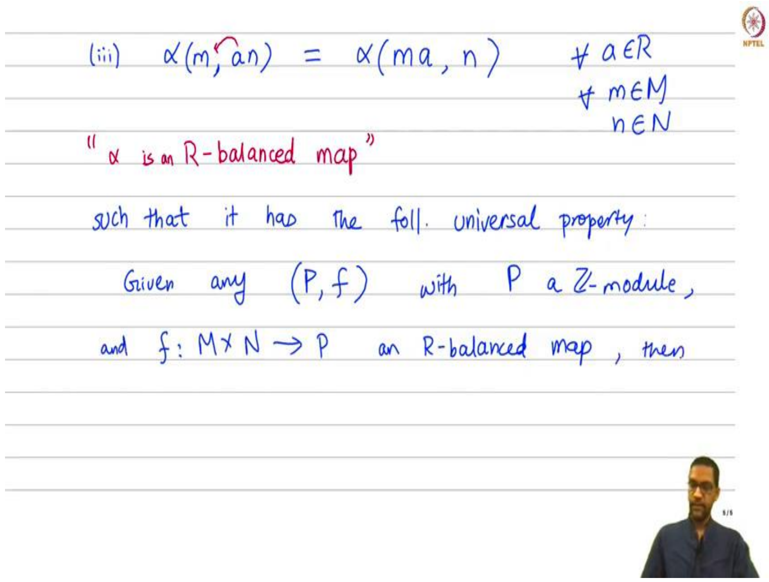

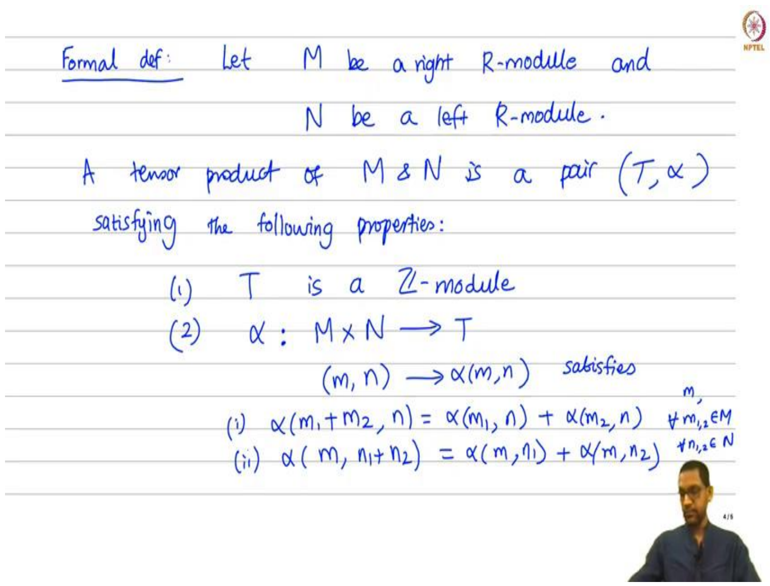

And the third property is what we just talked about heuristically which is that if I scalar multiply n by a scalar a, then that answer should be the same as right multiplying m by a and leaving n as it is. So, this is now true for all scalars a from the ring R and for all m and n for all m in M, n in N. So, this third property is the new notion. The other two are similar as before. So, this here again this is for all m1, m2 in M as expected n1, n2 in N and also, I should say for all m, m1 m2 in M and n, n1, n2 in N.

Now, so this alpha satisfies these three conditions. This third condition or maybe the set of these three conditions is called, so we say that alpha is an R-balanced map meaning it is bilinear in the sense m1 plus m 2 and n1 plus n 2 will give you the expected answer, but this new property here that this a can sort of be moved across from the second component to the first component.

That is sort of this notion of being balanced. So, in this terminology what we are saying is a tensor product of m and n is a pair T, alpha. T is a Z-module, alpha is an R-balanced map from M cross N to T, such that satisfying a certain universal property.

So, this R-balanced is what replaces just bilinearity when R was Z we only required alpha to be bilinear when you take an arbitrary R-module, you know arbitrary ring R then we also, require alpha to satisfy this third property. Such that it has the following universal property, what is the universal property? It says that given any such pairs, given any pair P, f with satisfying the same three properties P as Z-module, f an R-balanced map and f an R-balanced map then there exists, so let us draw the diagram now.

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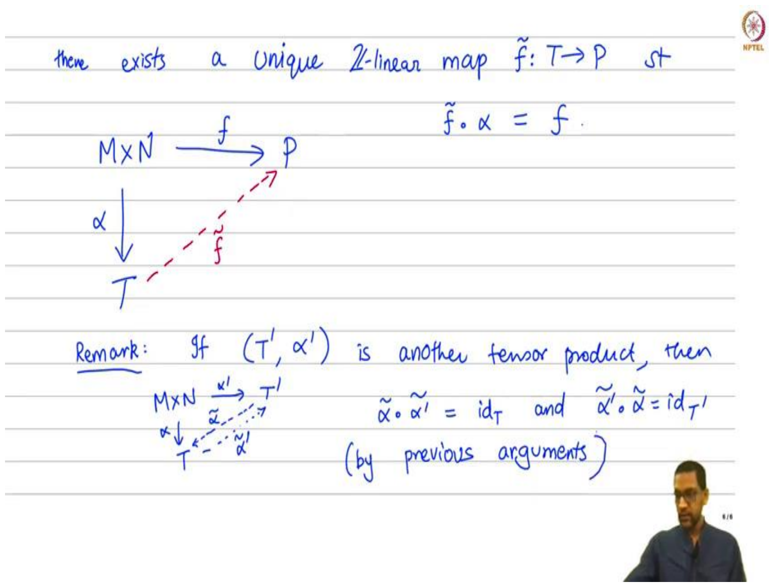

Then there exists a unique map. So, how does the diagram go from M cross N to T? That is my tensor product. Now what I am saying is given any P and any f any R-balanced map. I can find a unique map from T to P there exists a unique and what kind of map. Now T and P are both Z-modules or they are abelian groups So, there exists a unique Z-linear homomorphism or homomorphism of Z-modules, Z-linear map f tilde from T to P such that this diagram commutes f tilde composition alpha gives me f. So, you can see it is quite parallel to the earlier story of tensor product of Z-modules.

The only new ingredient is this new notion of being a balanced map. So, many things are, now that you have seen how to construct tensor products for Z-modules. It is very easy to see how to do things here. So, let us make various remarks, so observe firstly, remark as before you can think of all this as, you know in terms of categories and functors that this is some sort of certain initial object in a suitably defined category and so on.

But the key point here is that by similar arguments to what we did before, if I have another tensor product, if T prime alpha prime is another tensor product, then that map which you get, then the following is true that there is a unique isomorphism between them. Then look at this M cross N to T prime to T alpha alpha prime. So, by the universal property there is a map alpha prime tilde in this direction. Similarly T prime is a universal object meaning it is also a tensor product, it has the same universal property.

So, you can also, show that there is a map alpha tilde in the other direction. So, these were the same arguments which we gave for the case of Z-modules and one can show that their compositions are identities. Then alpha tilde composition alpha prime tilde is just the same as alpha prime alpha is the identity on T and by the same arguments as before, by previous arguments. So, what it means is that the tensor products if they exists are unique up to a unique isomorphism.

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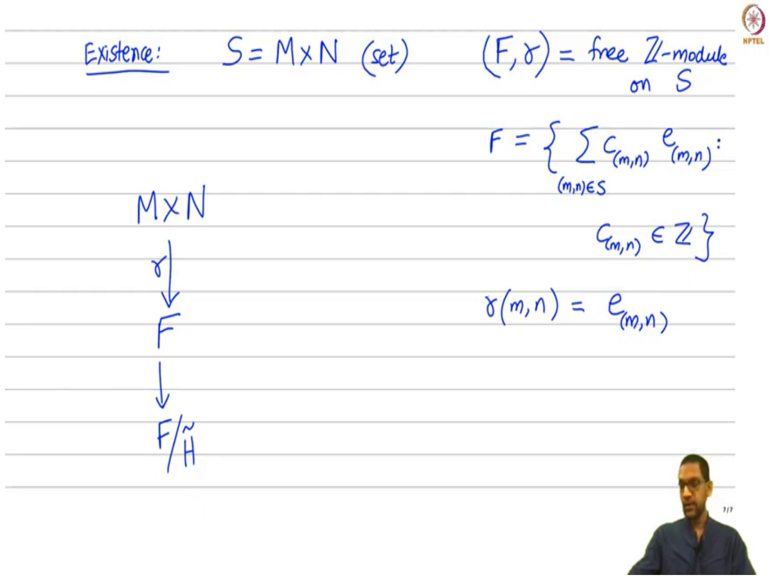

Now, let us just show existence and the proof of existence is more or less the same as we did in the case of Z-modules which is we proceed in two steps. We first look at the free module and then we look at the quotient module. These two same, very similar construction, so existence let us do the same thing. Let us construct, take the set S to be M cross N. Think of it only as a set, no additional structure. Let us take F to be the free module So, maybe F, gamma if you wish is the free Z-module on S.

So, recall what that means, well that just means the elements of F look like, the sort of linear combinations if you wish c m n e m n. Now e m n are m ns come from the set S, meaning from the cross product and I am only taking the free Z-modules. So, this is still elements of Z. So, any element of F can be written uniquely in this manner and we also, have this map. So, from M cross N this map gamma to F.

What does gamma do? Gamma just takes the ordered pair m, n and sends it is to that particular indicator function as we called it e sub m n and now the next step is to go modulo, a certain subgroup and F is an abelian group. I have to go modulo of subgroup. So, if you recall how this subgroup was constructed in the case of the tensor product of Z-modules.

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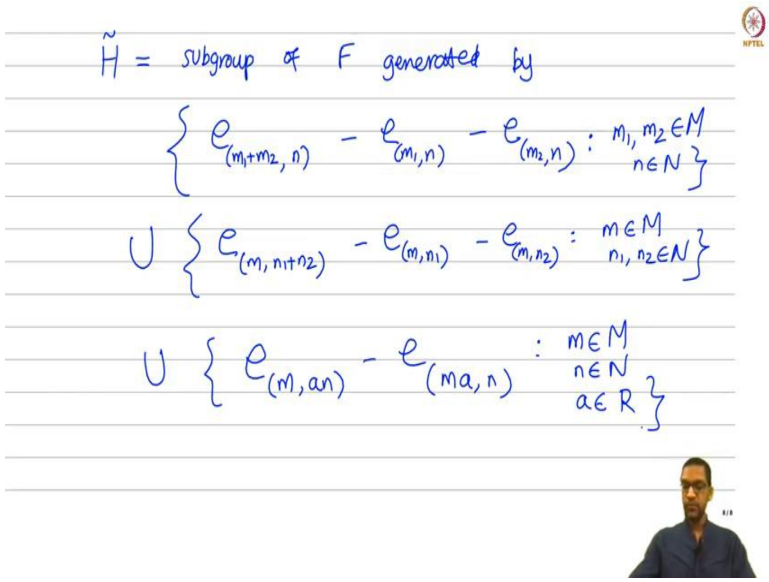

Let us just do a similar thing, what is H tilde? H tilde is the subgroup of F generated by the set of, you know by the following elements, elements of the following form take, generated by the following subset if you wish. Take all elements of the form e m1 plus m2, n minus e m1 n minus e m2 n, what other element should I take? So, this is for all N, union a similar thing with N and finally, so recall these were the two things you needed to satisfy the first two properties, but now we have this R-balanced condition So, we also, throw that in.

So, this is for all m in M, n in N and now a comes from the ring R. So, this is a key thing that now the elements of the ring R are now involved in this. So, this is a new, you know this H tilde is different from the previous H. In the sense that that only involved the first two subsets this also has this third additional set of elements. So, as before it is certain, it is a subgroup f is abelian group is a normal subgroup. So, I can look at the quotient.

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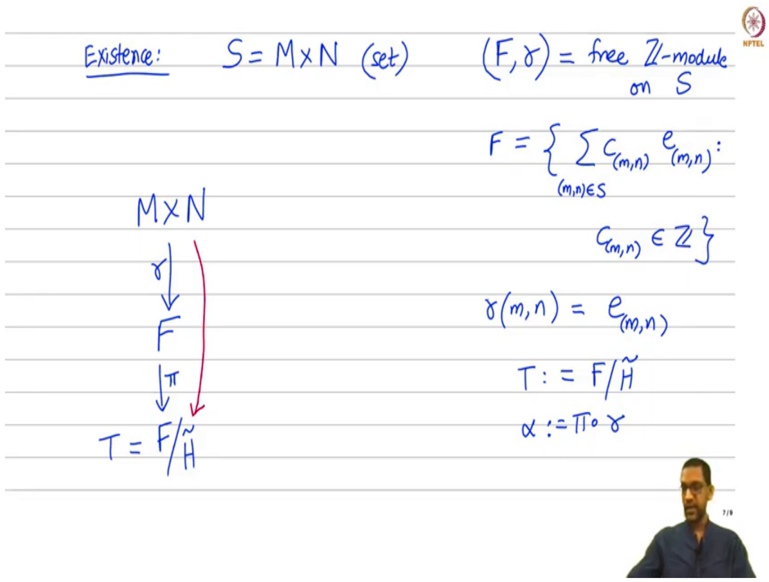

Which is what we did here. So, this is the quotient group and that is going to be our, so this is the quotient map which sends F to its quotient. So, this quotient is finally is going to be our tensor product module T. So, we define T as the quotient and we define this map alpha from M cross N to T to just be the composition. So, define alpha to be the composition, pi composition gamma. So, now the claim as before is that this satisfies all the properties that we needed to satisfy.

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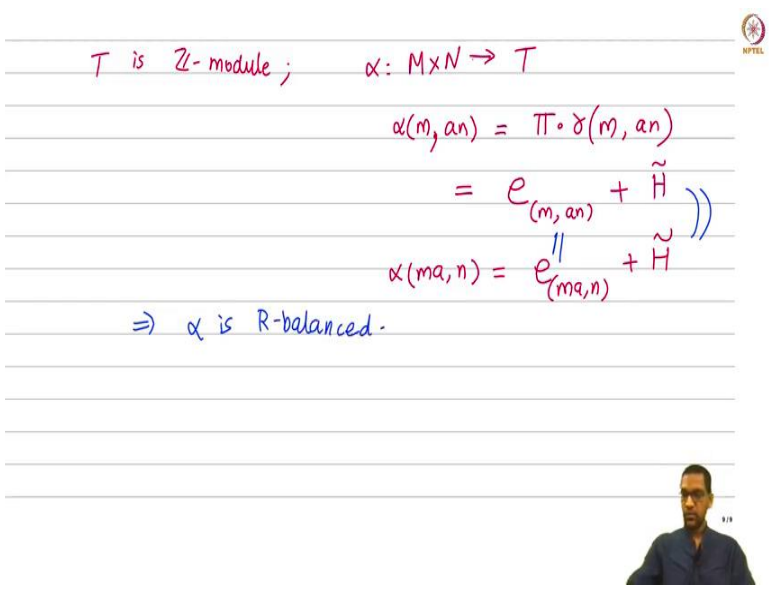

So, observe as before that T is a Z-module is obvious by construction. Claim is alpha satisfies the R-balanced condition. Alpha is a map from M cross N into T. Why does alpha satisfy the R-balanced condition? So, I will just take the third, the new axiom, the earlier ones are similar. So, what is alpha acting on m, an well by definition, this is just pi composition gamma of m, an and if you see what that is, this is e of m, an and pi just means you take its coset modulo H tilde.

So, maybe I will just write it like that plus H tilde. On the other hand if you compute ma, n by the same token this is just e of ma, n plus H tilde but observe these two cosets are equal why because the difference between these two elements m an minus ma, n. This difference is in H tilde So, these two cosets are exactly equal and the same sort of argument works for the other two axioms. So, alpha is indeed R-balanced. Now by the previous sort of argument we will again show that this has the requirement universal property.

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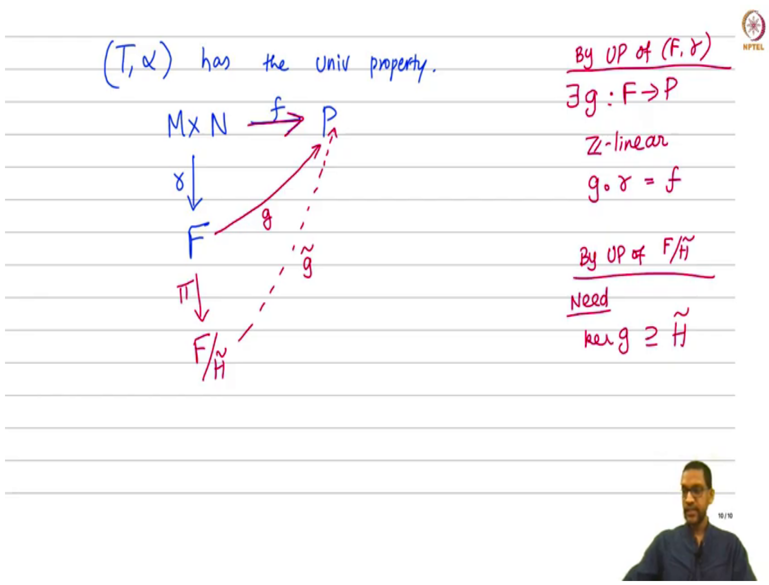

So, let us check that T, alpha has universal property. So, what does that mean we have to take any pair M cross N to P and then construct a unique map from T to P. Now we proceed like we did in the previous case for Z-modules, what did we do? We said okay. First we know that the free group F the free Z-module that has a universal property which is that given any set map from M cross N to a group P to a Z-module it gives you a unique map from F to P.

So, this is a map. Let us called it g. So, there exist a map g from F to P. This is a Z-linear map satisfying g composition gamma is f and this is by the universal property of free abelian groups or free Z-modules. Now second universal property that we need is we go to the quotient F mod H tilde and we said that the universal property of the quotient was the following that if this map g has a property that it is kernel contains H tilde then there is a unique map from the quotient to P.

So, by the universal property of the quotient to do this, well we need to first check, before we use the universal property I need to first check that the kernel of g contains H tilde. In other words I should check that every element of H tilde, the generators, it is enough to check the generators. I need to check the generators of H tilde all belong to the kernel. So, let us pick various generators So, I will just check it on the third kind of generator the new one that we are dealing with, the other two verifications are like before.

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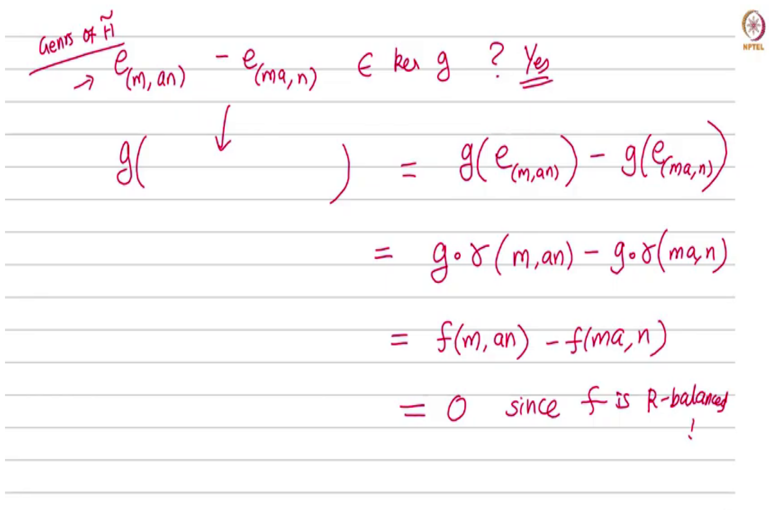

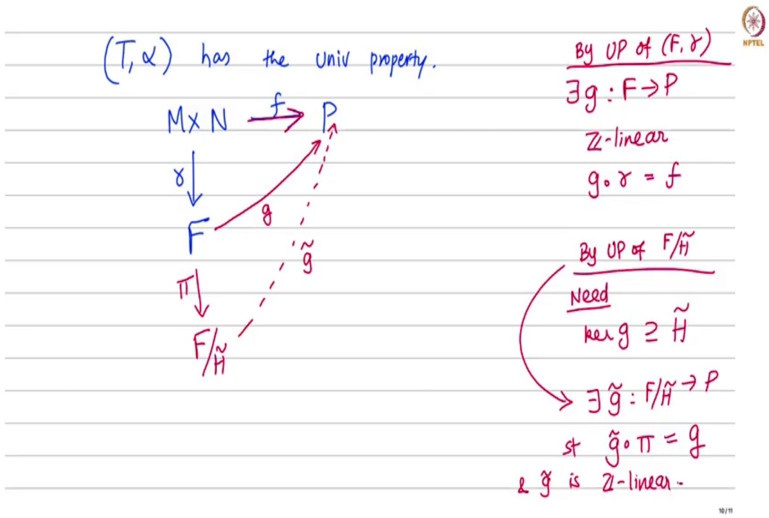

So, let us take the generators what is one of them m, an minus e ma, n. So, what am I doing here? I am trying to write out the generators of H tilde and this is one kind of generator. Now I need to check that this belongs to the kernel of g, is this true or not? Let us check, let us apply g to this. Now g is a group of homomorphism. It is a Z-linear map which implies that this is just going to be this. But then what is this? Well this is just g composition gamma of m, an minus g composition gamma of ma, n. Now what was the property of g? Well, g gamma was f. So, this is f of ma n minus f of ma, n, but observe that is 0 because f was given to be R-balanced.

We are only trying to show the universal property relative to pairs P, F where F is R-balanced So, what this means is because F is R-balanced that element, this particular generator of H tilde in fact belongs to the kernel. So, what we have shown is, yes it does. Now because it does belong now the universal, I mean you have to check the other two generators similarly, the other two kind of generators.

So, this by the universal property of f tilde we conclude that there exists a map g tilde from f mod H tilde to P such that it is again a Z-linear map there exists and g tilde is Z-linear. So, all this is more or less a repetition of the arguments we gave earlier. So, now finally what does this mean?

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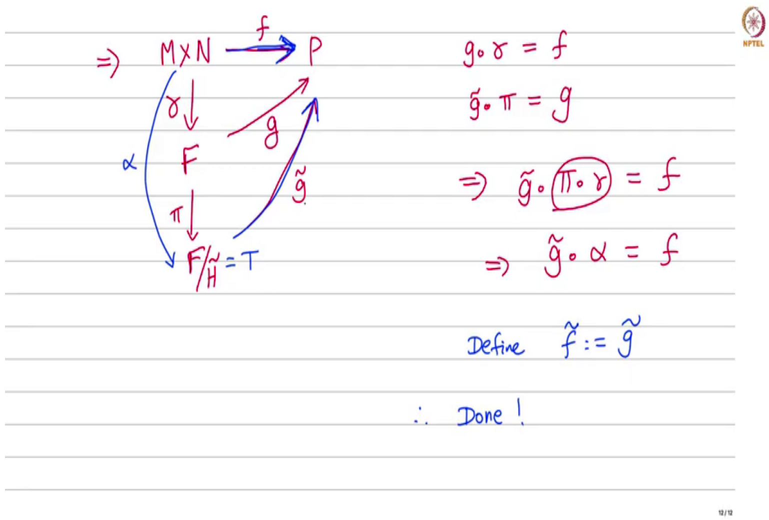

This means that we have shown the universal property, because I have two commuting triangles. So, on the one hand there is g, on the other hand there is, so this was F mod H tilde. So, there is this map g tilde, so what do we have? We have two commuting triangles. So, g composition gamma was f, g tilde pi was g like we did earlier you substitute g and conclude that this is f.

And of course this means pi gamma was exactly our alpha. So, therefore, g tilde alpha is f and that is exactly what we need to prove. So, g tilde alpha this map here is alpha. So, we need to show that you can find a map. So, I mean like we did earlier. So, let us call g tilde as f tilde now So, let us just define f tilde as just the map g tilde. So, we have shown that this outer triangle also, commutes because the two smaller inner triangles commute the outer triangle commutes.

Therefore done, so we have shown that the pair T, alpha, so this is T satisfies the universal property and again, I mean not quite, we have not fully done it because we have to show such as g tilde is unique, but that again is the exact same proof as before So, I am just going to leave that part as an exercise.

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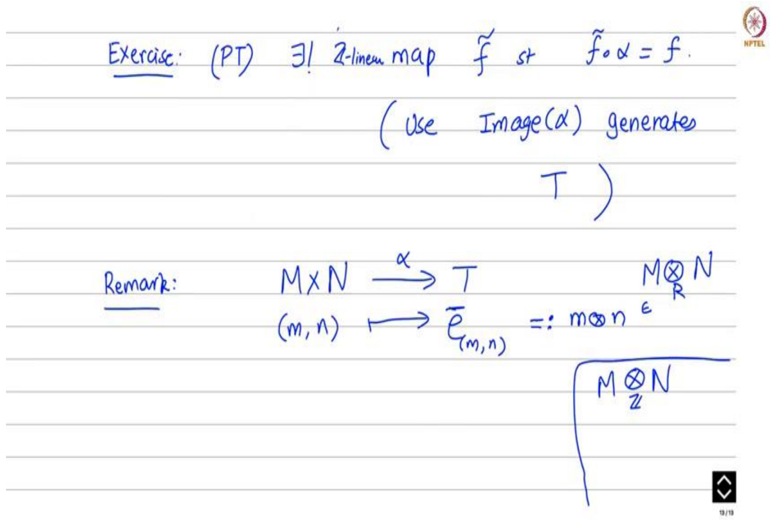

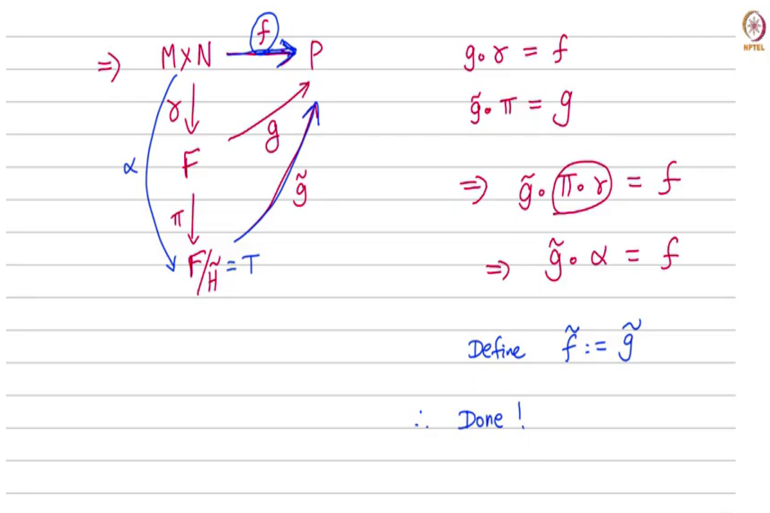

Exercise, prove that you know there exists a unique such map f tilde. There is a unique Z-linear map f tilde such that f tilde composition alpha is f and again the argument is similar to what we did before that the image of alpha, so again you have to use the same facts as before that the image of alpha generates the subgroup, I mean generates the group T as a Z-module.

So, very good, so we have shown the existence of the tensor product and again remark on notation just like we had in the case of Z-modules, if I have M cross N to T this map alpha. So, alpha remember was e m n bar. So, we give this a name we now call this again little m tensor little n, just under the same notation as before but we understand from context what tensor product we are talking about. So, if we are thinking as, I mean we are looking at the tensor product of m and n thought of as right and left R-modules then this m tensor n just refers to this particular element which I constructed just know for you by going modulo H tilde.

So, observe that you know there is also, you know the need to be careful here, it is because of the following, if M and N are R-modules I can ignore the R-module structure think of them only as Z-modules if you wish, just the abelian group. So, I can also construct another object called M tensor N Z, this is a different object, this is, if you wish if you, this has a different universal property which is with respect to the maps where I just take f to be the usual bilinear map without being balanced, for such maps you get this universal property.

So, that object is M tensor N over Z, that was over previous construction, whereas if you look for M tensor N over R that is a different object that has a universal property with respect to the R-balanced maps and its construction is different and in fact you can sort of see, how these two constructions mesh together.

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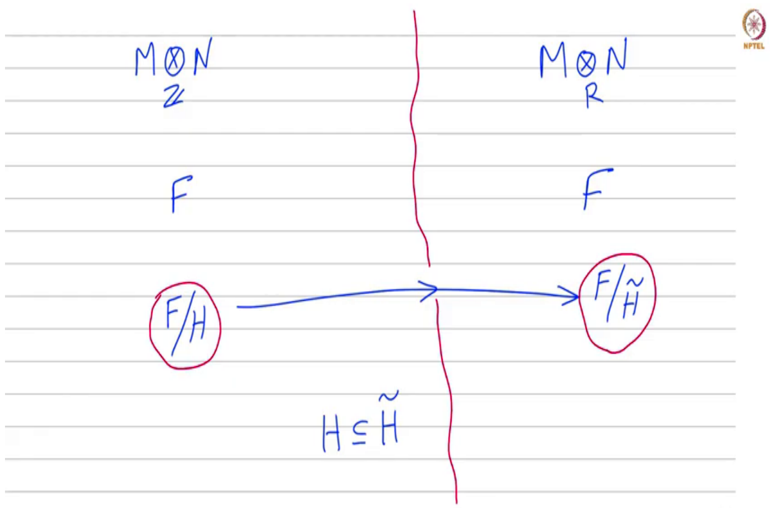

The earlier construction when we talked about M tensor N and Z versus M tensor N R, if you notice the construction proceeded along the same lines I first took the same free abelian group F. But the difference lay in what I went modulo, in this case I go modulo H which was only the subgroup generated by the first two kinds of generators whereas here I go modulo H tilde which als, has a third kind of generator.

So, observe H tilde is actually bigger than H, H is a subset of H tilde. So, if you wish as an abelian group if you go modulo, H tilde is actually larger than then H So, F modulo H to F modulo H tilde you get a map, you get a surjective map if you wish from just general principles about groups that G modulo a smaller subgroup versus a larger subgroup there is always a quotient map. So, this is called, the whatever, the third isomorphism theorem.

So, this is the, I mean we are not going to use any of this but just to keep these concepts separate in one's mind it is useful to know that you can actually think of M and N only as Z-modules and you know try doing the same construction but what you are doing there is constructing a different object with a different universal property and if you want to compare these two objects this is the relationships. That the Z-module that we constructed in the first instance actually has a subjective to map to the Z-module that we constructed in the second set.