Algebra 2
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Lecture 61
Problem session

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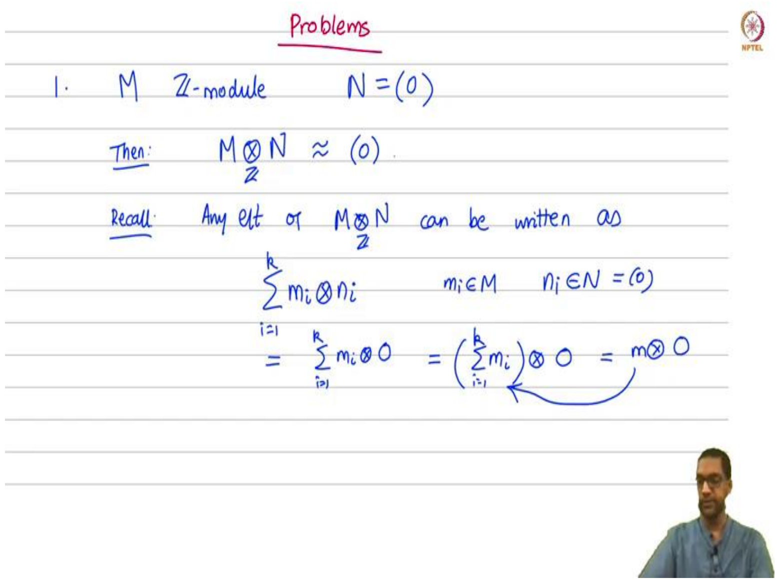

Let us do some problems or examples on tensor products. So, first problem. So, we define the tensor product of two Z-modules. So, now suppose I do the following I take M to be any arbitrary Z-module. But for N I will choose it to be the 0 module meaning it is just the singleton comprising 0. So, this of course is Z-module, the trivial group if you wish. Then the claim is that if I take the tensor product of M with N, then this is also just the trivial group. So, I claim that this is also just isomorphic to the trivial group. So, let us prove this. So, we just need to show that this tensor product has only one element.

So, what is the typical element of the tensor product, so recall this is one of the facts, every element has the following expression. Any element of M tensor N can be written like this. As a sum m i tensor n i or some finite sum let us say. Now m i comes from M, but n i comes from N which means the only possibility is that n i can be 0. So, what does this look like? It just looks like summation m i tensor the element 0.

But recall that this symbol, this tensor product symbol between M and N this was our alpha of M, N that we looked at last time and that was a bilinear symbol, meaning this is the same as summation m i tensor 0. Now, so let me just call this sum m tensor 0. m is just summation m i, so I will just give this a new name. Summation m i is M. Now this element m tensor 0, I claim is actually the element 0. So, let us prove this.

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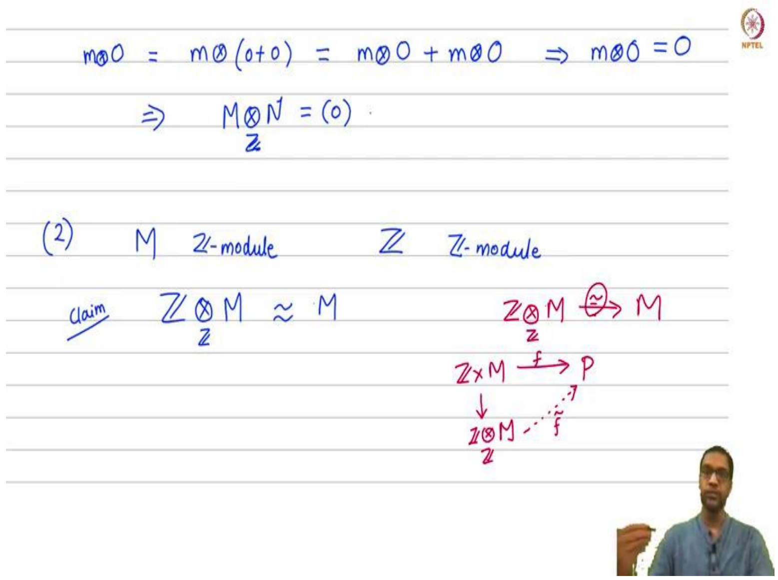

So, claim is that m tensor 0 is the additive identity of this group m tensor M, why is that? Well let us again use the bilinearity, so 0 plus 0 is 0. So, m tensor 0 is m tensor 0 plus 0 but now bilinearity tells me that I can split this into two pieces. So, m tensor 0 is the same as two copies of m tensor 0 which means m tensor 0 has better equal the element 0. So, that means what we have shown is that any element of the form m tensor 0 is just 0, therefore this module that we constructed the Z-module, it does not have any elements other than the 0 element.

Because we started with an arbitrary element and we showed that that element is just the 0 element., so that is the first problem. Now more generally, well let us take a second situation suppose M is a Z-module. Then consider N to be Z itself. I mean take the second module to be Z itself.

So, Z itself is a Z-module and what I am going to do is just going to look at their tensor products. So, suppose I take Z tensor m, so then I claim that what do I get? So, we already did this, if you tensor with 0 then you get just the 0 module. If you tensor with the module Z then what do you get? Well you just get back a copy of the original module.

The claim is that this tensor product is actually isomorphic to M itself. So, let us prove this claim and to prove this claim what we will do is we will use the universal property of the tensor product. So, observe what does proving this claim entail? You have to show that there exists and isomorphism from Z tensor M to M. This is what we want to construct, an isomorphism between these two guys and this sort of map from a tensor product to another module is given by the universal property.

So, recall that if somehow you could construct, so this is the universal property of the tensor product. If you have a bilinear map to some P then that gives you a unique map from the tensor product to P. So, if you are trying to construct a map from the tensor product to any module then what you should try and do is to somehow use the universal property of the tensor product.

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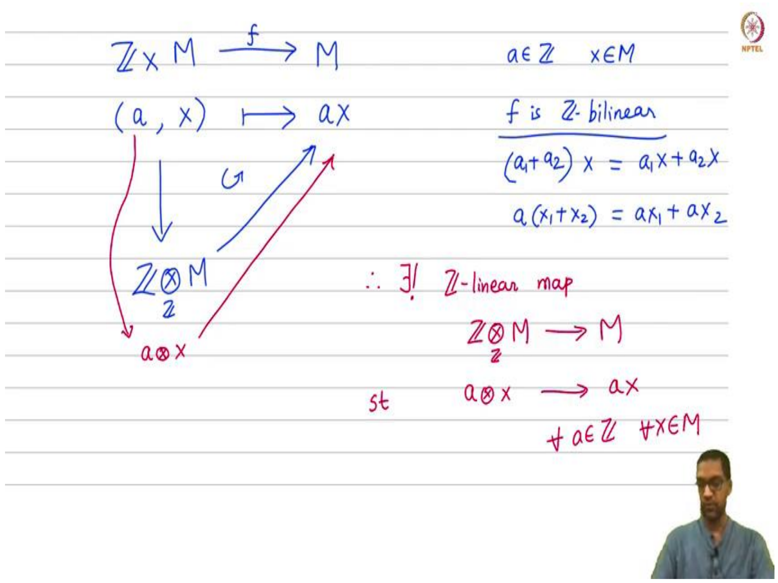

Let us do that in this case to see how it works. So, let me first start with the bilinear map from Z cross M and the target module here is M itself, so first let me try and construct a bilinear map. What should I take? Well it is, there is more or less only one thing you can do. Take an integer a, take an element x of M and just map it to a time x.

I mean M is a Z-module, so I know what scalar multiplication by an integer means. Now observe that f is bilinear, f is in fact a Z bilinear map, easy to check because if I, that is just the module property. If I take a sum of two scalars I get this or if I keep the scalar fixed and take a sum of two module elements, so I have Z-bilinear map.

Therefore by the universal property of the tensor product, I have a map from here to here and what does this map do? So, it makes this diagram commute. There exists a unique map we know that and what does this element a, x, like what does that map to in this? So, recall that was exactly what we called, so alpha of a, x is what we had denoted by the symbol small a tensor x.

So, this map does the following, it maps a tensor x to ax therefore we have concluded there exists a unique Z-linear map such that which takes a tensor x to x. So, this is just another way of saying it makes the diagram commute. Now claim is this map is an isomorphism. That is going to be our claim.

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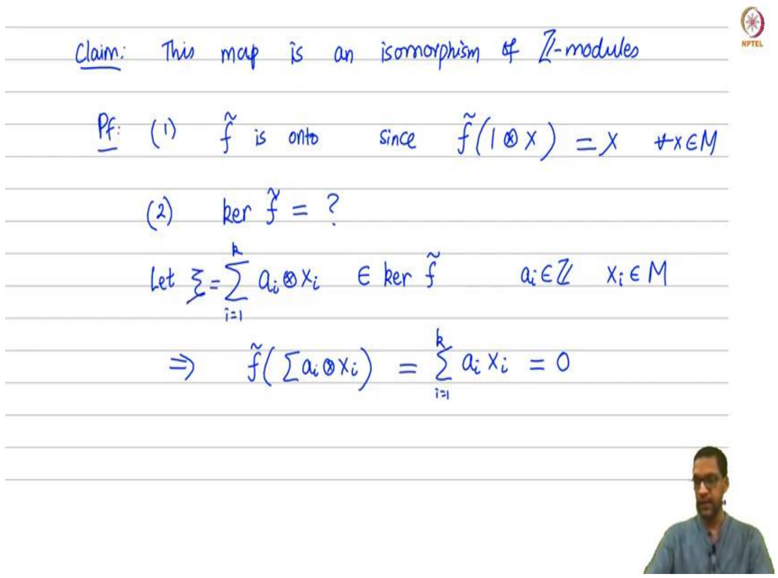

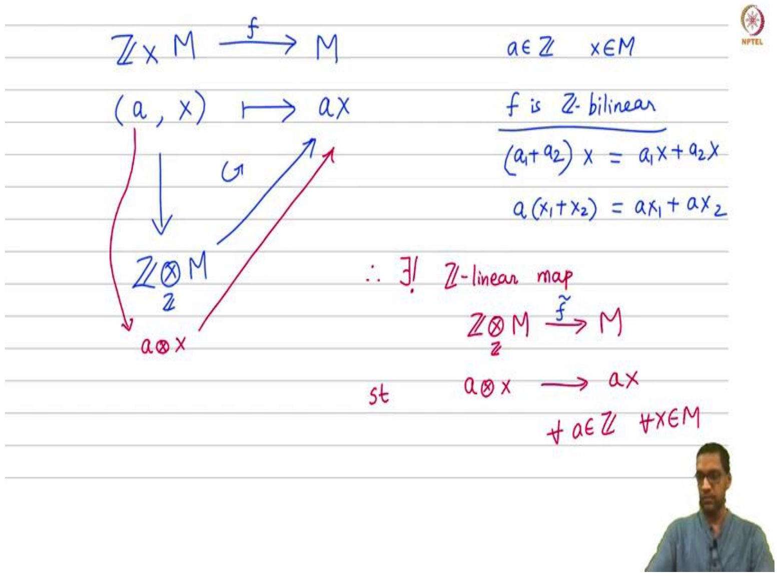

Claim is an isomorphism of Z-modules meaning it is an isomorphism of abelian groups. Proof we will show it is 1 to 1 and onto. So, first observe it's onto is clear. We will give this map a name f tilde. F tilde is onto because you just have to take a to be 1. So, observe property 1 f tilde is definitely a surjection.

So, definitely every element x is in the image. So, we just need to check that the kernel is 0. So, let us try to understand what the kernel looks like. So, let us take a typical element of the kernel. So, let some element of the tensor product belong the kernel, but this is again where we keep using this important fact that any element of the tensor product can always be written in this form.

Summation a i tensor x i and recall this was just another way of saying that those e m n bars or the image of alpha that generates the tensor product. So, let us pick a typical element, suppose this belongs to the kernel. Any element can always be written like this.

Now this belongs to the kernel means what, I apply f tilde to this, I should get 0. But I know what I get when I apply f tilde. It is just the product a i x i. So, suppose this is 0. Now observe the following that the original element that wrote down, so this is the conclusion, the summation a i x i is 0. Now this original element that I wrote down, so maybe I will call this something like xi belong to an element of the kernel.

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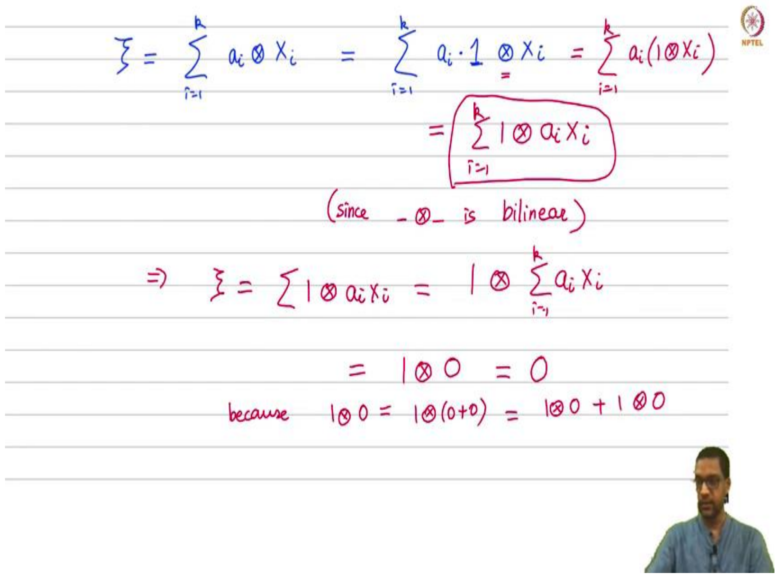

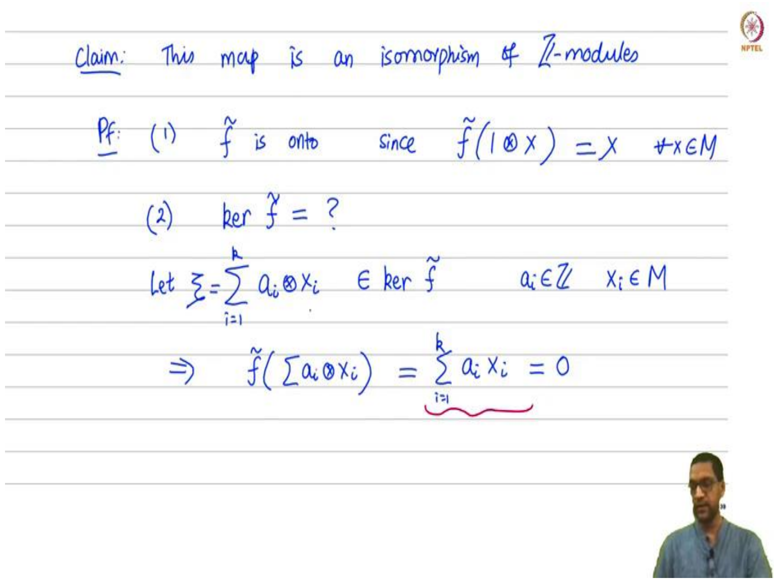

The xi can also be rewritten as follows. So this is summation, but recall the bilinearity also says the following. So, I will rewrite this as a i times 1, I will multiply it by 1 and this symbol here, this tensor product symbol, in other words, alpha, the map alpha is bilinear means that I can pull this a i out, so this is just nothing but summation. So, we had this property.

It is, I can either write it as a i times 1 tensor x i or as summation 1 tensor a i x i, these are all properties coming from the bilinearity, since this tensor product symbol is bilinear. When I put two elements there it is linear in each component. Now let us look at this last expression. The element xi has this alternate expression here as summation 1 tensor a i x i this means that xi equals, again I am going to use the bilinearity, so here we keep using the bilinearity in a very important way.

So, this is the same thing again by bilinearity and now recall this is just the element 1 tensor 0 because we have just shown that summation a i x i 0 that was our assumption if that, this element is in the kernel. So, therefore this element xi we started out has this other nice expression. It is just 1 tensor 0, but now recall that is exactly our earlier proof that 1 tensor 0 is actually the element 0 of this module.

Why is this 0? Meaning it is the additive identity, well because the same argument 1 tensor 0 is of the form 1 tensor 0 plus 0 and you can write it as a sum of 1 tensor 0, 2 copies. So, the same proof xi is just the additive identity so therefore what we have done? We started out with an element of the kernel and we showed that that element is just a 0 element. Therefore the kernel is 0.

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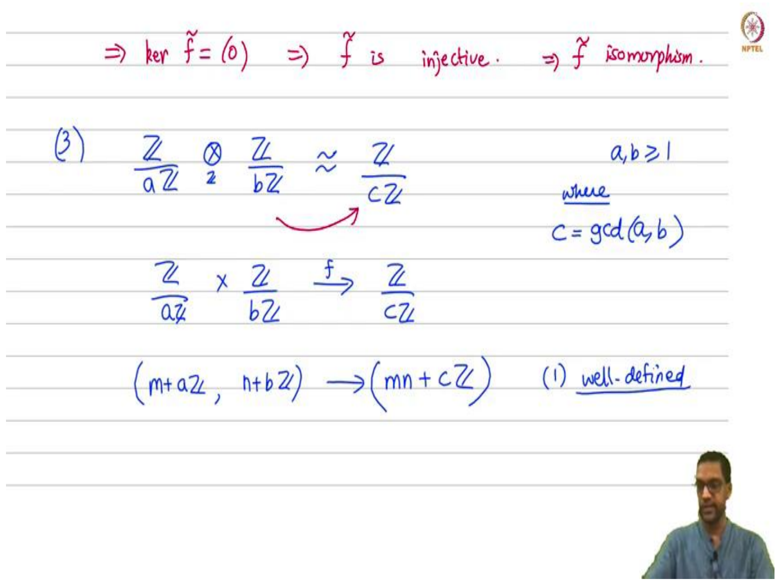

It implied f tilde is also 1 to 1. So, it is 1 to 1, I mean it is an injective map if you wish. So, it is 1 to 1 and 1 to map therefore it is an isomorphism, which is what we needed to prove. So, let us look at another problem, which is to show that to understand the tensor product of abelian groups. So suppose I take, of cyclic groups, if I take two cyclic groups Z mod aZ and Z mod bZ imagine a and b are some numbers, at least 1. So, I can always assume they are positive numbers.

Then I claim that there is again a cyclic group cZ, where c is just the gcd of a and b. So, let us prove this. So, again our strategy is going to be the same. We will try and use the universal property to construct a map from the tensor product to this other module and then we will show that that map is actually an isomorphism.

So, let us do this quickly. So, from Z mod aZ cross, so I need to start with the bilinear map, start with the map f, what should we do? Well let us take any two elements so what is the typical element look like m plus aZ coset, n plus bZ and what can we do this.

Well, let us just map it to mn plus cZ. So, I am defining a map like this. You need to always check when you define a map like this whether it is well defined, so property 1 that this is a well defined map, why is it well defined? Well if I change m or n to some or both maybe, to some m prime and n prime. Let us check well definedness.

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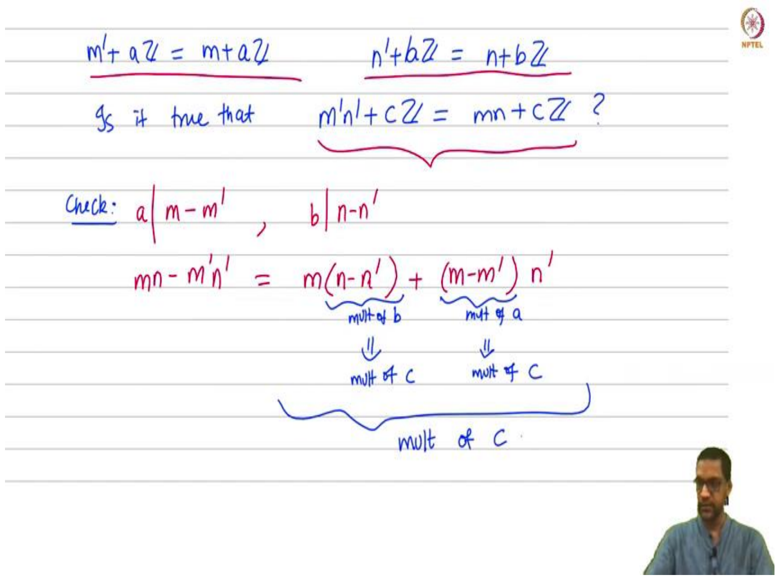

So, if I change, so suppose I change the representative to some numbers m prime and n prime then the question is if my final answer the same? Is it true that m prime n prime plus cZ gives me the same answer. That is what I need to check. Let us check what do I know. This tells me that m minus m prime is a multiple of a, it belongs to aZ.

So a divides this. this property tells me that n minus n prime belongs to bZ meaning it is a multiple of b. Now let us look at what we have to prove which is this guy. Let us take the difference, mn minus m prime n prime. We need to show that is a multiple of c, but we can rewrite it as follows, this is a little addition and subtraction of the same term. I have added and subtracted minus mn prime from both. Now observe this is a multiple of b, this is a multiple of a.

But observe that c is the gcd of a and b, so if something is a multiple of b it is definitely a multiple of c, a multiple of a for the same reason is a multiple of c, so this both terms are multiples of c, so then answer is again a multiple of c, which is what we needed to prove that mn minus m prime n prime is in fact a multiple of c.

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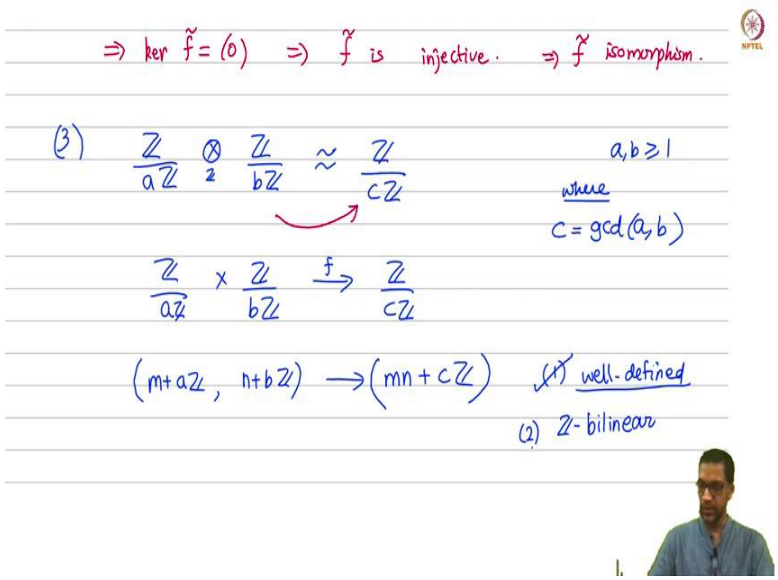

So, it is well defined. It is easy to see it is a group homomorphism. So, 2 group homomorphism or a Z-module homomorphism if you wish, maybe I will just say it like that, I should have said Z-bilinear rather, it is Z-bilinear. So, let us also prove that. I need to prove it maybe. So, let us show that it is Z-bilinear, so in each component it is linear.

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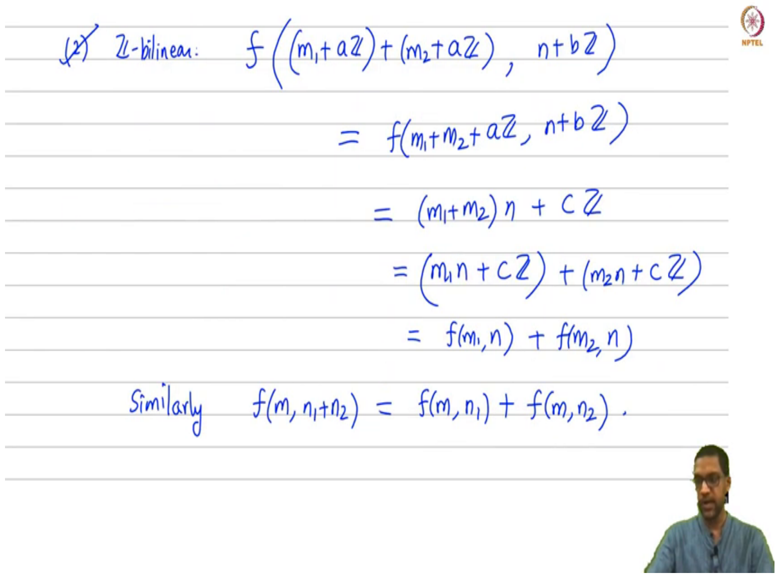

So, the second property Z-bilinear. So, if I look at m1 plus aZ what is f evaluated at, if I take a sum of two things in the first component what is the answer? Well by definition this is just f of, so this is how you do additions of cosets and this by definition is m1 plus m2 into n plus cZ, but observe that is the is same answer as m1 n but cZ plus, so it splits in to two pieces so that is exactly f of m 1 n. So, it is linear in the first variable similarly you can check the other, similarly check n1 plus n2 is again a sum of two things. So, it is in fact a Z-bilinear map so that is a second property.

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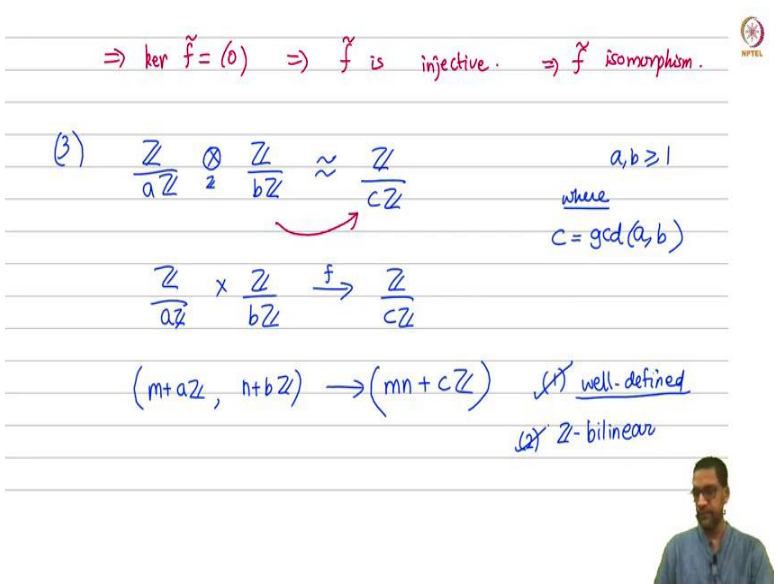

So, I have defined a well defined Z-bilinear map from Z cross aZ, Z mod aZ cross Z mod bZ to Z mod cZ and therefore by the universal property of the tensor product I will get a map.

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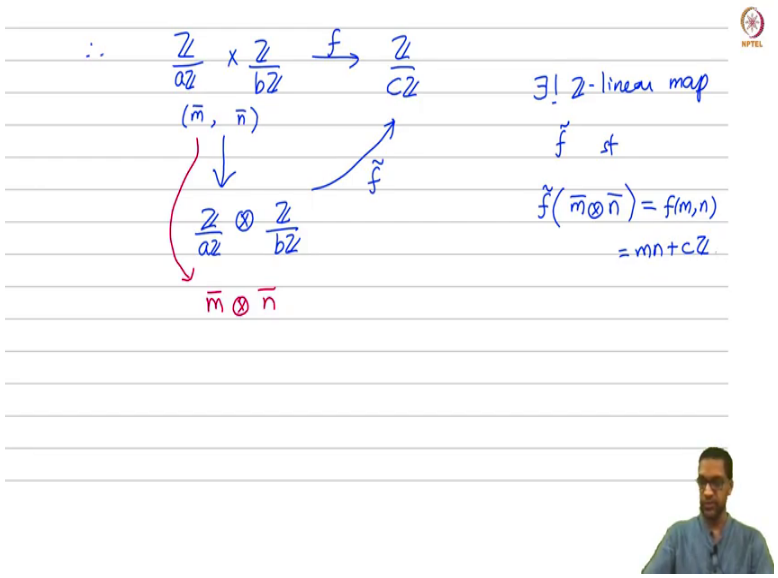

Therefore I obtain, so I have this map f. So now, so there exists a unique map which is homomorphism of Z-modules, so there is a unique Z-linear map f tilde such that the diagram commutes, meaning f tilde evaluated on, on what? Well if I take a typical element here, so let me just call this m plus aZ as m bar and n plus bZ as n bar, then that element here maps to m bar tensor n bar. So, now my property here is that m bar tensor n bar is map to f of m n, which was by definition mn plus cZ. Now claim is that this map is an isomorphism as before.

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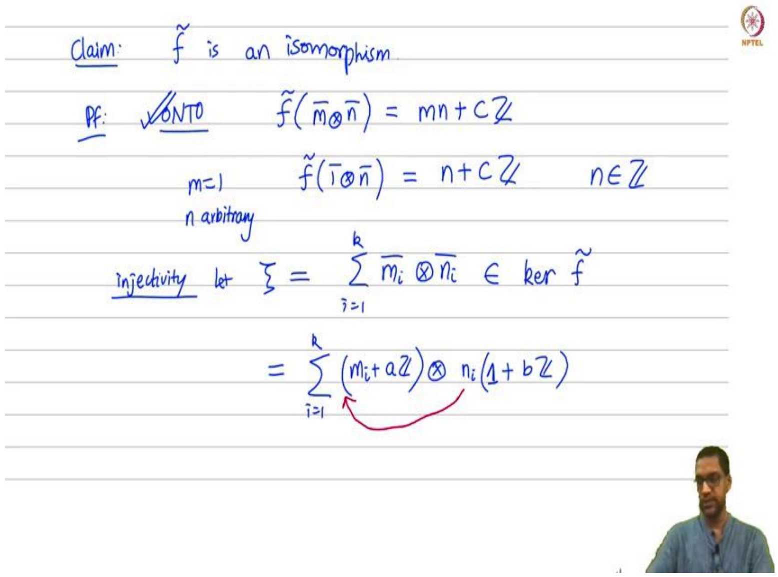

Claim f tilde is an isomorphism of Z-modules. Let us prove this quickly. So, what we need to do. We need to check it is onto, again that follows from just taking m equals 1, so observe that f tilde of, this was the definition. So, just put m equals 1 and choose all possible values for n. So, this is f tilde of 1 bar, m tensor n bar, so 1 tensor n bar is just going to give you n plus cZ and n now ranges over all possible values in Z anyway.

So, therefore every coset is of this form in Z mod cZ. So, this is onto for sure, choose n arbitrary then you are done. It is again the injectivity that requires more proof and the proof is almost along the same lines as what we gave for the previous problem which is let us take a typical element in the kernel.

So, let xi which is summation m i bar tensor n i bar belong to the kernel of this map. Now what does this mean? Well as before, let us just first rewrite it in this case, so this is summation, so what is m i bar? It is m i plus aZ and n i bar is n i plus bZ, but I can think of it as follows. I just take the particular coset 1 plus bZ and I multiply is by the scalar n i. That is the Z-module structure on Z mod bZ.

So, I can just think of it as scalar multiplying the coset 1 plus bZ by the integer n I, the scalar n i and now again we use the bilinearity of this tensor product symbol meaning the bilinearity of that map alpha which defines a tensor product, which says that this scalar here can be pulled completely out of the tensor product or equivalently you can push it in to the other component.

So, I can move in some sense the n i from the second component to the first component. So, let us do that this is by bilinearity.

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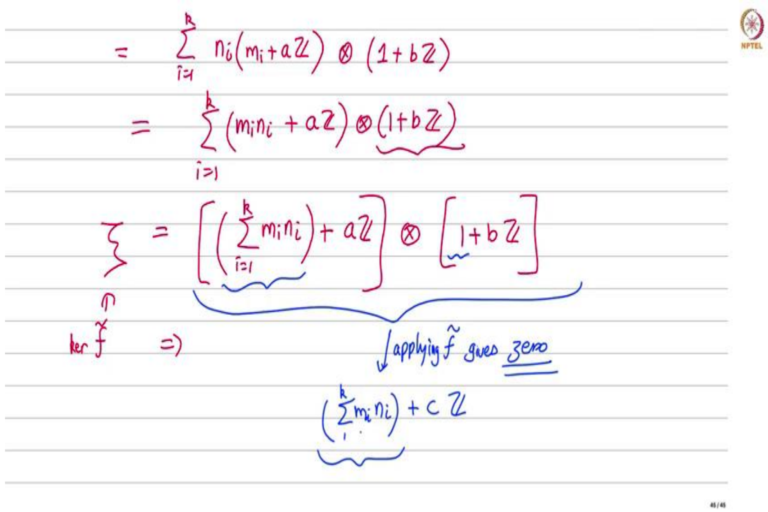

Therefore this is the same as summation n i multiplying m i plus aZ. So, this example is to give you some facility with manipulating expressions involving tensor products so 1, but what is this? This is just summation m i n i plus aZ 1 plus bZ and observe that the second comprehend is the same for all terms, for all sum and so I can again use bilinearity make this into summation of these cosets m i n i plus aZ. Look at this coset of a tensor this coset.

So, what I have done is sort of rewritten my element xi which was in the kernel in this way and now I will use the fact that xi is in the kernel. So, xi belongs to the kernel means if I apply f tilde to this answer I should get 0, so let us do that. So, you take this, you apply f tilde. If you do that you will get 0. Applying f tilde gets 0.

Now let us apply f tilde. So, what is the definition of f tilde? If you go back and see you are supposed to multiply these two elements. In this case it is just summation m i n i in to 1 plus cZ that was the answer. So, this is 0, that is what we have concluded, so xi belongs to kernel, therefore it means that element here has to be a multiple of c meaning it is in cZ.

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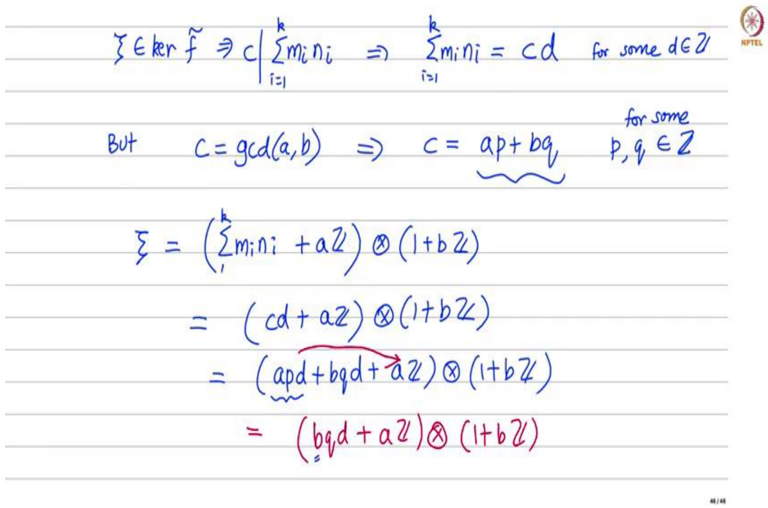

So, conclusion that it is a multiple of c, but observe, but c, what is c? c was the gcd of a and b which implies that if you recall what this means you can write c as some linear combination like this. For some elements p and q in Z. The gcd of two numbers is always expressable as a combination of these.

So what does this mean? So, this summation m i n i is of the form c into some number d, it is some multiple of c. So, let us rewrite xi as follows. So, xi was summation m i n i plus aZ, which I will write as cd plus aZ but c itself can be rewritten in terms of as and bs. So, I will use this expression now. So, what will I do? I will write this as apd plus bqd plus aZ.

Now observe that this apd is already a multiple of a. So, that already belongs to aZ so that term I do not need to worry about. So, that is already in here. It is already in aZ, so there is just bqd plus aZ. tensor 1 plus bZ, but now comes the interesting observation. So, observe this first term has a b in front of it. So, I will rewrite it as follows.

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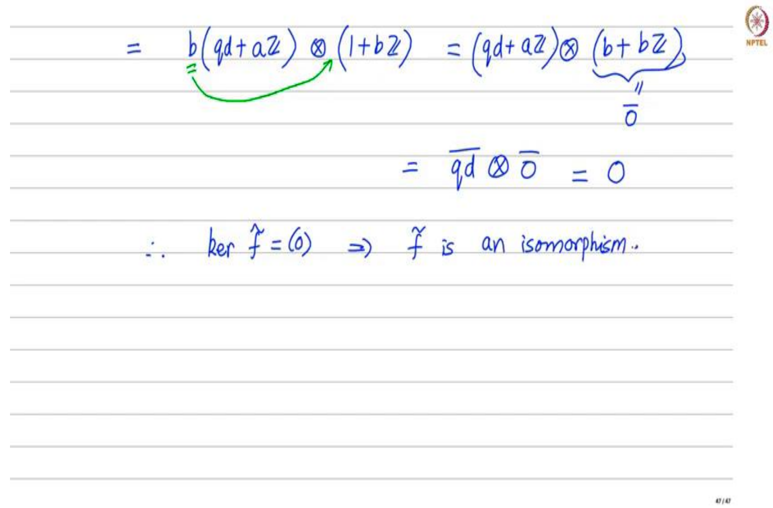

I will think of it as b multiplying qd plus aZ. So, it is as if I am taking this coset and multiplying it by b tensor 1 plus bZ. Now again I will use the bilinearity, what does bilinearity tell me? It tell me that if I have this scalar integer in front I can completely pull it out or I can, on the other hand I can also push it to the second component.

So, here I will just push it to the second component. This is there for qd plus aZ tensor b times this 1 plus bZ, but what is that? That is b plus bZ, but observe b plus bZ is just 0, because be belongs to bZ. So, this again is nothing but the 0 coset and now we are back to the same. I will call this qd bar again, 0 bar and now we have already seen that if you put a 0 in any one of the two components then using the 0 plus 0 equals 0 argument you can show that this is just the 0 element.

So, almost the same sort of idea as before, but just a little bit more complicated, therefore again the kernel of f is 0 therefore f is isomorphism, f tilde is an isomorphism. So, this is an interesting exercise because it says that, I mean this tensor product thing is slightly non-intuitive.

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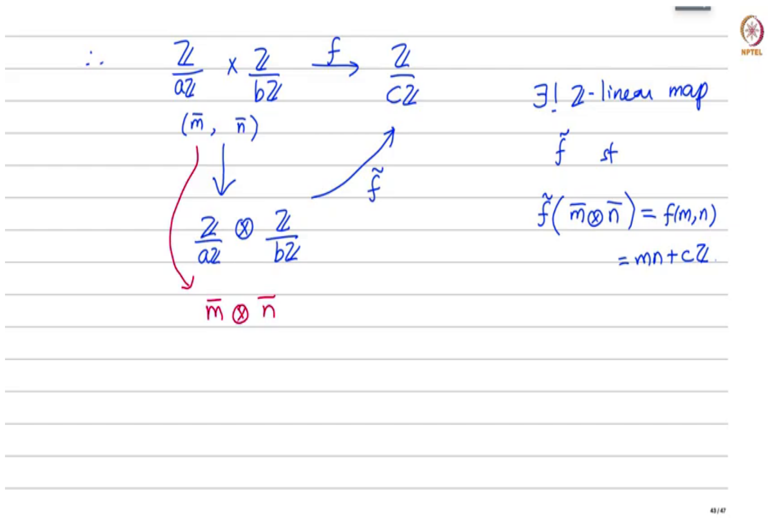

We started out with these two cyclic groups, Z mod aZ and Z mod bZ and what we constructed somehow is sort of a smaller cyclic group if you wish, whose size is equal to the gcd of the concises.

So, it is a sort of interesting gadget and to some extent what you need is really a lot of practice, manipulating these bilinearity relations and trying to understand how one can construct maps, what these moving scalars from one side to the other, things like these are all part of the toolkit, when you want to work with tensor products. so you should try and get a lot of practice doing these things.