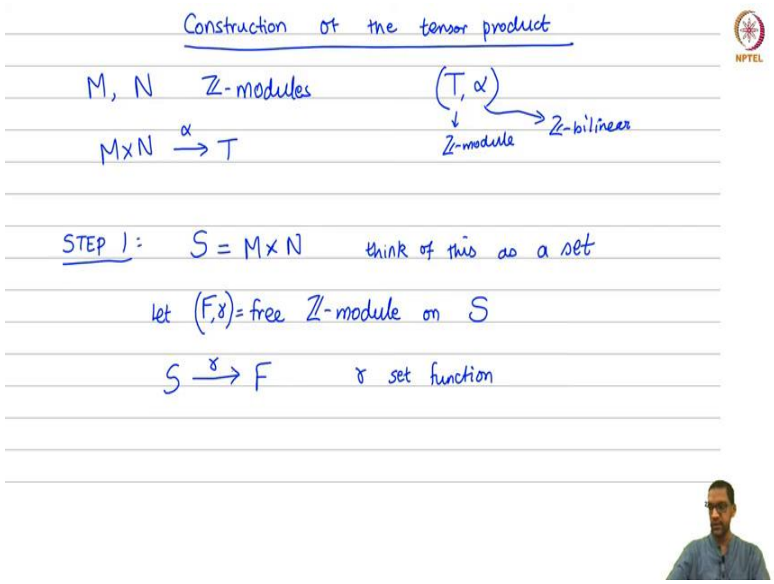
Algebra 2
Professor. S Viswanath
Institute of Mathematical Sciences
Lecture 61
Construction of the tensor product

(Refer Slide Time: 00:15)



Let us Construct the Tensor Product. So, we will show the existence by explicitly constructing the tensor product of two given modules M and N. So, recall starting point again, I need to have M, N two abelian groups as Z-modules.

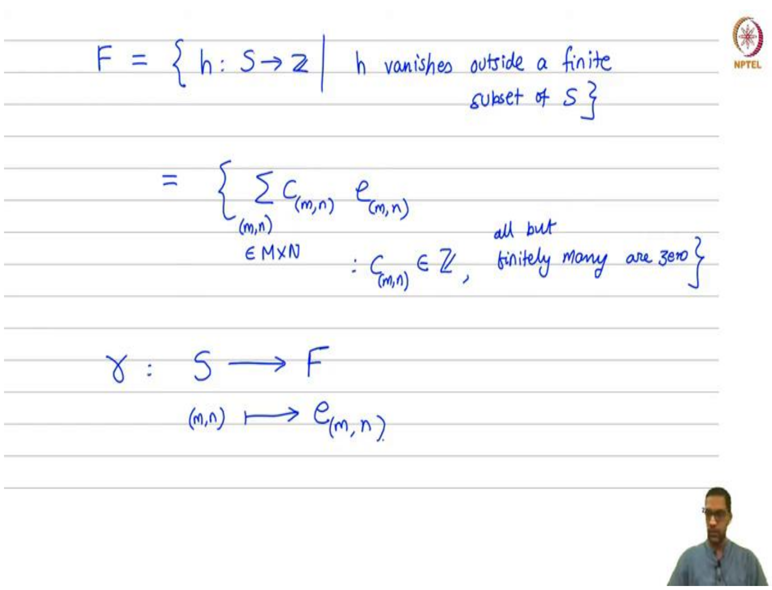
Now what was the tensor product? The tensor product was a pair T, alpha, now what is T? It was a map from M cross N, where T is, T is a Z-module and alpha is a map from M cross N to T. So, T is a Z-module and alpha is a map which is Z-bilinear. We need this such that there is a certain universal property which is satisfied relative to, if you take another such pair P, F then there is a map from T to P which makes that diagram commute.

So, that was the definition but let us start out by trying to construct this Z-module T. So, there are a couple of steps involved in this construction. The first step involves the free module that we talked about. So, let us do the following. Let us take define the set S to be M cross N. Think of it as a set, think of this only as a set, no additional structure, ignore the fact that there is an addition operation on M, there is an addition on N and so on, forget all that, just think of it as a arbitrary set.

And let us define F to be the free abelian group or free Z-module. So, let this be the free Z-module on this set S. In fact, if you recall the free Z-module was actually a pair. F, gamma so how did we define the free module, well it was, what was gamma? Gamma is just a map of sets from S to F just an arbitrary set function.

Such that there was a certain universal property with respect to such pairs given any other group and a map from the set S to that group, any other Z-module, any other abelian group then there is a map from F to that group. So, that is what we looked at last time. So, what we will do is we will just take the free Z-module on S, look at this map gamma, now in fact we constructed the free module explicitly so recall what is the free module, therefore what are the elements of F look like.

(Refer Slide Time: 03:17)

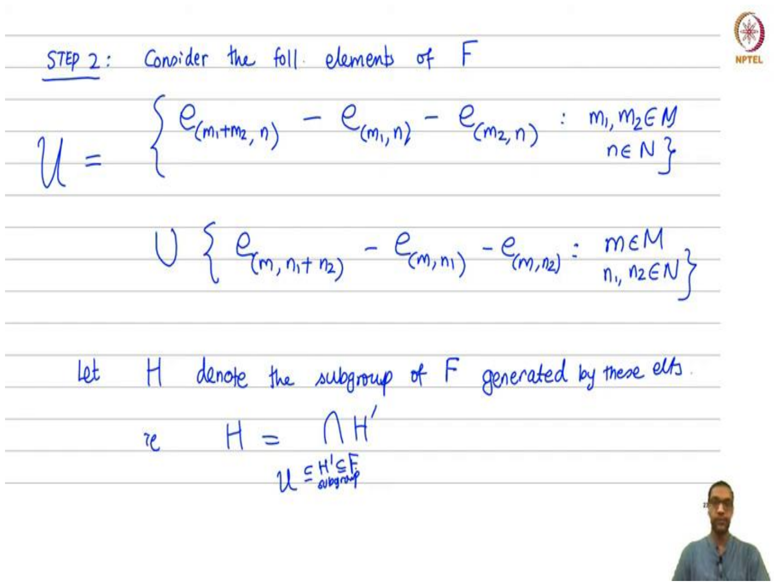


Recall that we constructed F as the set of all functions of finite support on S. So, this was all, you could think of it as functions from S to the integers such that h vanishes outside a finite set and but on the other hand we also wrote it in terms also, we said every element of S has in fact a unique expression as follows, so you can write it as, so in this case the set S is all ordered pairs m, n.

So, let me denote it like this where m, n ranges over the set S in this case which is M cross N. Where the C is C m n. So, we should say such that C m n is an integer and finitely many are non-zero, so only finitely many non-zero.

We should say all but finitely many are 0. So, this was the construction of the free module and this map gamma that we constructed, so given a set S there is this map to the free module on S which is the following, take any element S, this case the element m , n of S, you map it to corresponding indicator function e m n or think of it like a basis vector if you wish, it is1 only on m, n and 0 everywhere else. So, this was the map gamma. So, first step we construct the free module on the set S.

(Refer Slide Time: 05:04)



Now step two in the construction involves the other thing we looked at last time, the other construction which is that of a quotient group. So, consider the following elements of the free module. What are these? They are elements of the following form, so let us look at e m1 plus m2, n so this is an element of F. Now F is an abelian group, so I can look at this subtraction, so it is a Z-module. So, I can look at this element. So, this is some element of S. So, consider the following elements of F. So, this is an element of F.

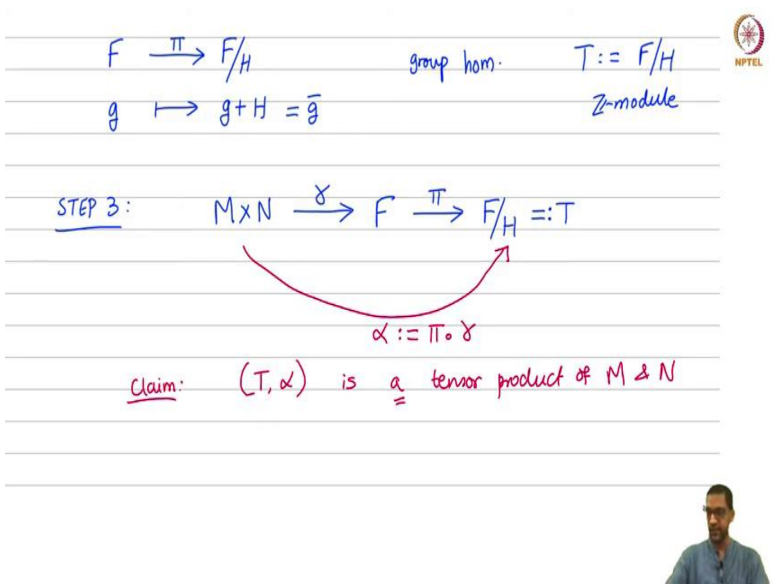
So, take all these elements where m1 m2 come from M n comes from N, union the same sort of elements in the other component. Now m comes from M, n1 n2 come from N. So, I have constructed a collection of elements and what I will do is, I will take the subgroup of F generated by these elements.

So, let H denote the subgroup of F generated by these elements which means it is the smallest subgroup which contains these elements generated by these elements i.e. H is, well what is that?

It is the intersection of all subgroups H prime where H prime is a subgroup of F and H prime contains this collection of elements.

So, maybe we should call these elements something. So, let us call this the set U and H should contain the set U, H prime should contain U. So, it is an intersection of all such subgroups H prime. So, I should say H primes are subgroup. Now of course here everything is in abelian group, F is abelian and so any subgroup is automatically normal. So, what we can do is look at the quotient group.

(Refer Slide Time: 07:25)



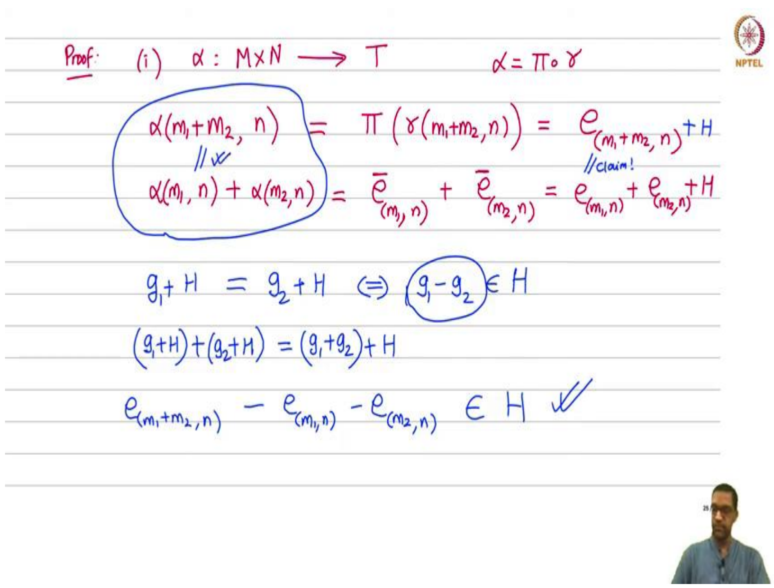
So, we will look at the following take F and look at F mod H that is the quotient and there is a projection map, so what is this? Given any element of F you just look at its coset. So, here everything is written additively, it is in abelian group. So, instead of writing it as g H instead of the multiplicative notation I will just switch to the additive notation. So, I will call this coset as g plus H. So, which is what you usually write as g bar the coset of g. So, this is of course a group homomorphism.

Now let us sort of put these two things together in step three. So, what have we done so far on the one hand from M cross N, thought of as a set. I got a set map to the free group. Now from the free group I have a group homomorphism. So, pi remember is of course a group homomorphism to F mod H. Now I will call this as T this is going to be our tensor product. So, this is my definition, that this the Z-module T. So, observe, so let us define T to be just the abelian group F mod H. So, it is an abelian group, in other words it is a Z-module.

And I need to define a map in order to say that this is the tensor product. I need to construct a map from M cross N to T. The map is just the composition these two maps. So, I will call this composition as alpha. So, what is alpha? It just denotes the composition of pi and this map gamma.

So, our claim now is going to be that T, alpha is a tensor product of M and N. So, I say a tensor product but recall we already showed that if it exists the tensor product is unique up to a unique isomorphism, but we need to first show that this satisfies the axioms or the universal property of a tensor product. So, let us first check what do we need? T is Z-module is all right, but we need to check that alpha is a bilinear map. So, let us verify our claim.

(Refer Slide Time: 09:46)



So, here is the proof of the claim. First thing we need to check is that alpha is in fact bilinear. So, what is alpha? It is a map from M cross N to T. Now alpha is just pi composition gamma and it is sort of strange because gamma is just a map of sets. Pi is a group homomorphism, but we are claiming that at the end, after you compose these two this alpha has this property of bilinearity in the two variables.

So, let us check that this is true so what does bilinearity require us to check? We need to take m1 plus m2 in the first component, n in the second component and let us see what that goes to. According to this definition this is pi of gamma of this, which is just well, gamma of this is just the element e m1 plus m2, n and pi is just mapping it to its coset modulo H.

On the other hand, we need to check whether this is the same as alpha m1 plus n, m1 n plus alpha m2 n. We need these two things to be equal so, the question is are these the same. So, let us verify, so I will compute the other side, this by the same token is e m1 n bar this coset plus e m2 n bar.

What do these cosets really mean? Cosets of course mean that I look at this plus H. So, the question is when are these two equal? Are these equal? How do I check? Now how do you check that two cosets are the same? So, suppose I give you a coset g1 plus H and I need to check whether this is the same as g2 plus H.

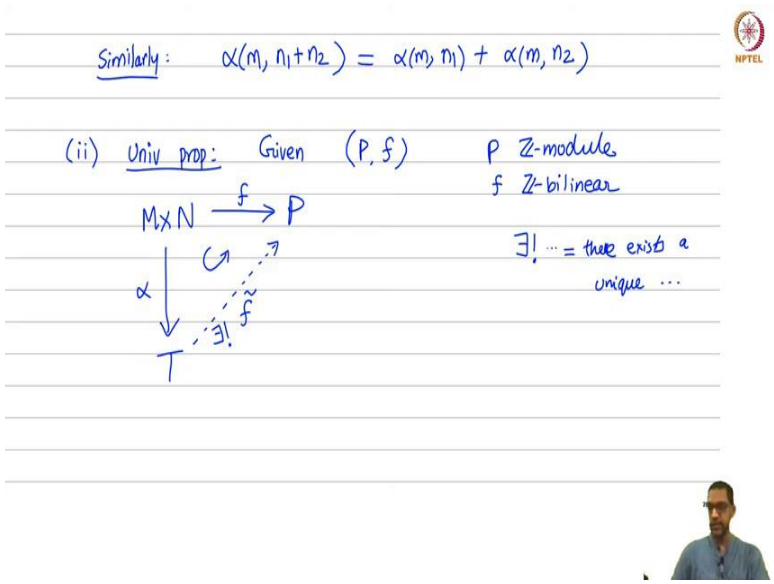
Then recall that this is just the same as saying, well I will use again the subtraction here instead of inverse, g1 minus g2 belongs to H. This is exactly what it means for two cosets to be the same, so here we just check similarly. So, the addition by the way, so when I say e bar of this plus e bar of this I am trying to add two cosets, but recall that is just the same as. So, let us also simplify this a little bit more, this is nothing but, so how do you add two cosets? So, for example what is g1 plus H plus g2 plus H. What is the addition operation on the coset space, on the quotient? It is just you add the representatives.

So, here again I do the same thing, I get e m1 n plus e m2 n, this is an element of F and it takes, it is coset modulo H. So, let us check that these two answers are the same. I claim that they are in fact the same. This is the claim. So, let us check. So, to check it I need to take that difference and check that the answer is in H.

So, let us compute what is the difference of the two representative? So, this again is that, is actually this plus H. So, let us write out the two expressions, on the one hand it is this and I need subtract from it whatever is on the other side e m1 n minus e m2 n and I need to check whether this element is in H. But recall this is exactly one of the generators of H.

I mean that is the reason why H was chosen in this particular way. This was of course, this is definitely in H because it was one of the generators of H. So, similarly the other one, so this is done. So, the claim is true. So, in other words the first bilinearity in the first variable is in fact true that this equality holds. We have proved this, but the same sort of argument with the second component is also true.

(Refer Slide Time:13:52)

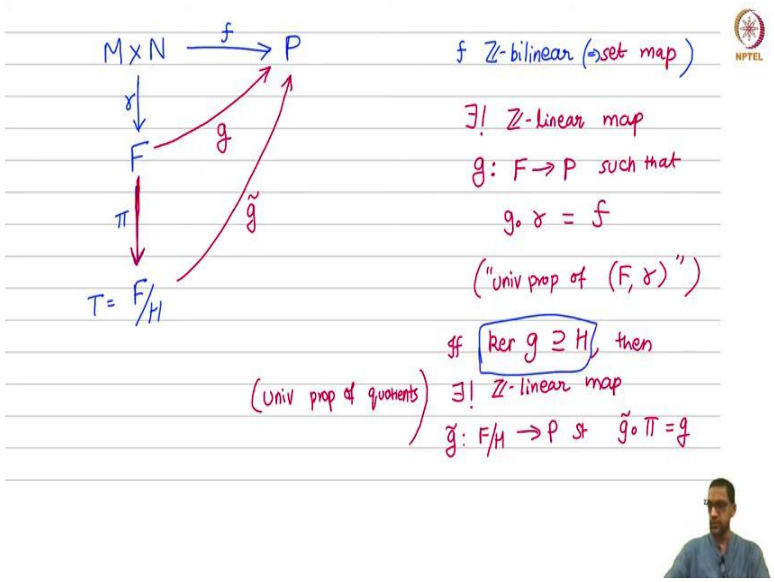


Because that is the other generator of H, the other kind of generator. So, this is the same as, same proof. So, we have shown that alpha is bilinear, but what is the second thing we need to do? We need to check whether this pair T, alpha has the universal property of a tensor product. Let us check the universal property. So, what we need to do? We need to start with some pair P, f.

Given P f, P is a Z-module and what is f? f is a map from M cross N to P which is bilinear. f should be a bilinear map Z-bilinear. Given these two I claim that we can do the following, we can construct, so what do I want to do I want to construct a map from T to P. I want to construct a Z-module homomorphism, Z-linear map f tilde.

Such that this diagram commutes that is f tilde composition alpha gives me f and this should be unique also, so there exists f tilde and this is unique. So, this notation here just means unique, there exists with an exclamation just means there exists a unique whatever it is that we want to say. So, the uniqueness is denoted by the exclamation, so let us show that this is in fact true. For this we need to go back to the definition of T. So, recall T was constructed in two steps. So, we will also construct this map via those intermediate steps. So, let me do this.

(Refer Slide Time: 16:03)



So, on one hand I have a map from M cross N to P, f is actually Z-bilinear but for a start f is, think of f only as a map of sets. Why do I want to do that? Because the very first step in the construction of the tensor product involved using the free module. So, before I bring in T let me first do F. Now look at this diagram F is Z-bilinear, but at the moment let me not worry about that, f is just a set map, I will only think of it like that. I will come back to the Z-bilinearity.

So, it is a set map therefore by the universal property for free modules, I know that there exists a unique map from F to P. So, this is a unique map, so maybe we should call it f, maybe something else, so it is not to confuse with the tilde there. Let me call it f bar, okay, so there exists a unique map f bar, maybe we will just call it g, there is a unique map g.

What sort of map is this? This is a map of, it is a homomorphism of groups. There is a unique Z-linear map g from F to P such that the diagram commutes. What I have used here? Is just the universal property of the free group. So, this is the universal property or the definition if you wish of the free group F, gamma, the free abelian group or a free Z-module.

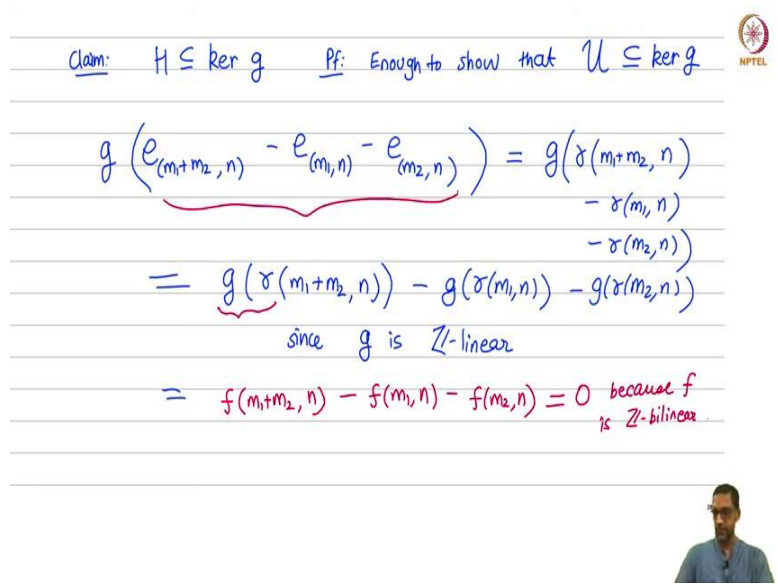
Now step two, we will use the universal property of the quotient. Now recall F mod H is what is T, this is a projection map. Now forget about the top part of the diagram let us only focus our attention on just this portion. So, I have this map, just look at the arrows in red, I have the arrow pi and I have the arrow g and what was the universal property of the quotient F mod H?

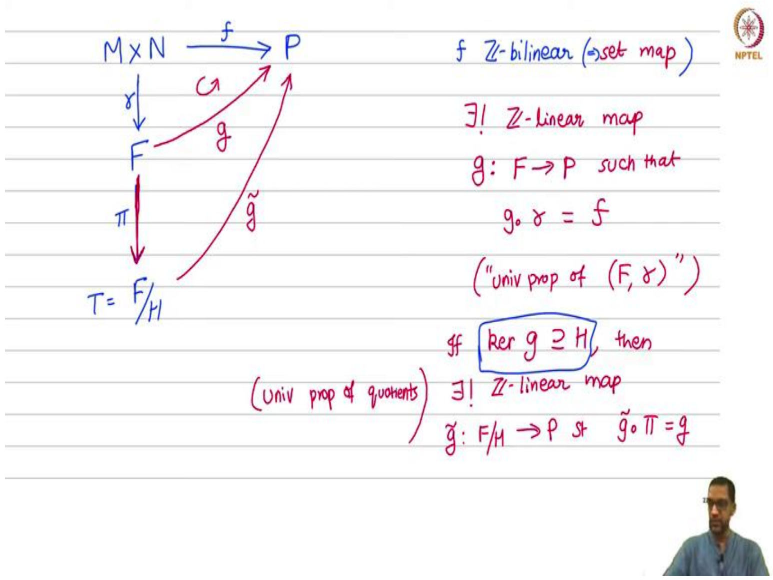
If I have a group homomorphism from F to P which, whose kernel contains H, then I know that there exists a unique map from F mod H to P. So, now step two is if the kernel of g contains H then we know the following, then there exists a unique map, maybe which we now call g tilde. This is a unique group homomorphism, so in this case it is, everything is an abelian group.

So, I can think of it as a Z-module morphism, there exists a Z-linear map, g tilde from F mod H to P such that this diagram commutes, meaning the diagram in the red, g tilde composition pi gives me g. This was by the universal property of, so this last bit is the universal property of the quotient group.

Now let me show that, I claim that the kernel of g certainly contains H. Now how do I prove this? Now that is why, I need to go back and use the fact that it was, f was not just an arbitrary set map, it was actually more than that. Of course if it is a Z-bilinear map in particular it is a set map. But I am going to use the Z-bilinearity to show that the kernel of this map g contains the subgroup H. So, the claim is this is true. So, let us prove this.

(Refer Slide Time: 20:01)





So, claim kernel of g is certainly contains H. So, H is inside the kernel of g, remember how I have to prove this, H is generated by a bunch of generators. If I show that the generators are all there inside kernel of g, then it automatically follows that H itself is inside the kernel of g, because the kernel is a subgroup and H is the intersection of all subgroups which contain that collection of generators.

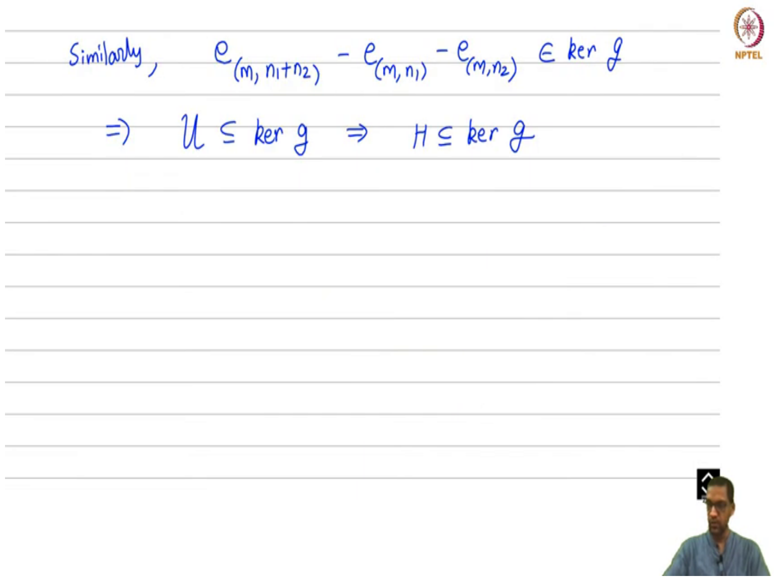
So, proof enough to show that the generators belong, enough to show that the generators which I called U is contained in kernel of g. Now let us prove this. So, what is a typical element of U? So, let us show it for one kind of element, so look at e m1 plus m2, n minus e, so this is a typical element of U and let us show that it is inside the kernel of g. In other words if I apply g to this I will get 0.

Now how do I do this? Well observe that, So, what do we know about g, g composition gamma is F. Now what are these es? They are just gammas. So, if you see, what is this map, this is just gamma evaluated on the ordered pair m1 plus m2 minus gamma of, just by definition, what does gamma do? It just maps an ordered pair to e of that ordered pair.

Now g is however a group homomorphism. So, I know that g is Z-linear. So, this splits into three pieces. So, this is g of, gamma of, since g is a group of morphism, since g is Z-linear, but now observe g composition gamma, which is what I have here, this is exactly F that was the defining property of g.

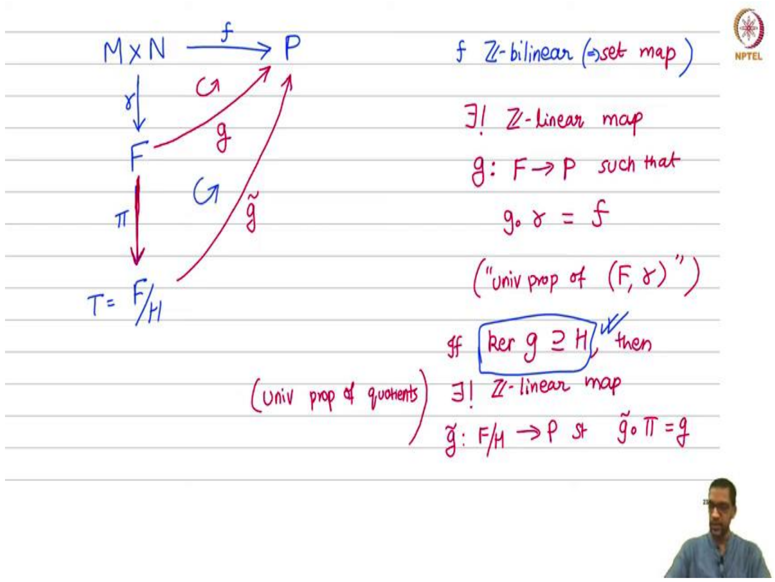
It makes this diagram commute, so g composition gamma was f, so I will replace all the g gammas by fs. So, this is nothing but f of m1 plus m2, n but then that is 0 because f is bilinear, because f was bilinear. So, what this implies then? So, we have showed that this generator belongs to the kernel similarly you can show the other generator.

(Refer Slide Time:23:09)



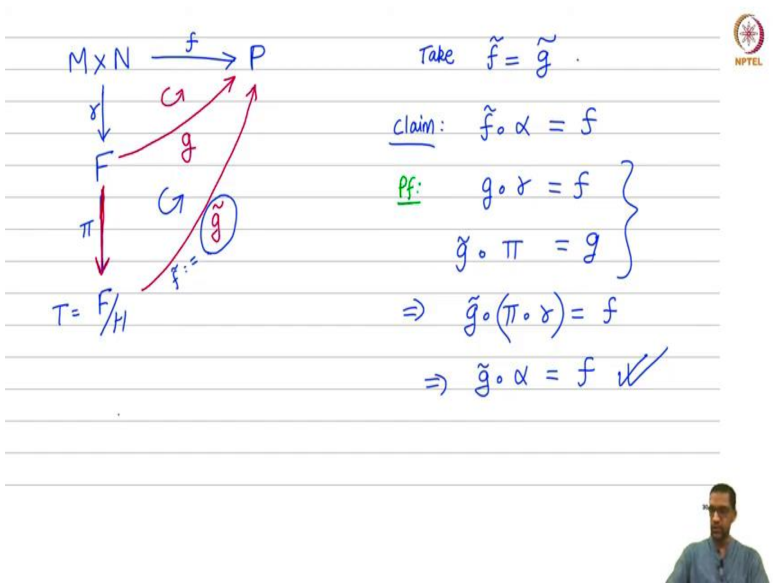
So, similarly, similar argument shows that the other generator e m, n1 plus n2 is also in the kernel of g. So, we have managed to show this that all the generators belong to kernel g and as I said before if the generators belong then the subgroup they generate definitely also is a subset of the kernel. And now therefore we are done, therefore by the universal property of quotients what does it imply? It implies that there exists a unique map.

(Refer Slide Time: 23:54)



So, this, we have shown that kernel of g contains H, this is true, therefore there exists a unique Z linear map which makes this diagram commute. So, what we have finally established is that this or the existence of all these maps. So, let us just copy this. So, we have shown now that.

(Refer Slide Time:24:20)



So, we have constructed starting from F we constructed this map g by using the universal property of the free abelian group, then starting with g we constructed g tilde by using the universal property of quotients and now this is going to be our final map. So, the claim is that g tilde is what we need, so I mean that is what we called as f tilde. So, we take f tilde to be this final map g tilde. So, take f tilde to be this map g tilde. So, because I mean f tilde is what we needed to construct.

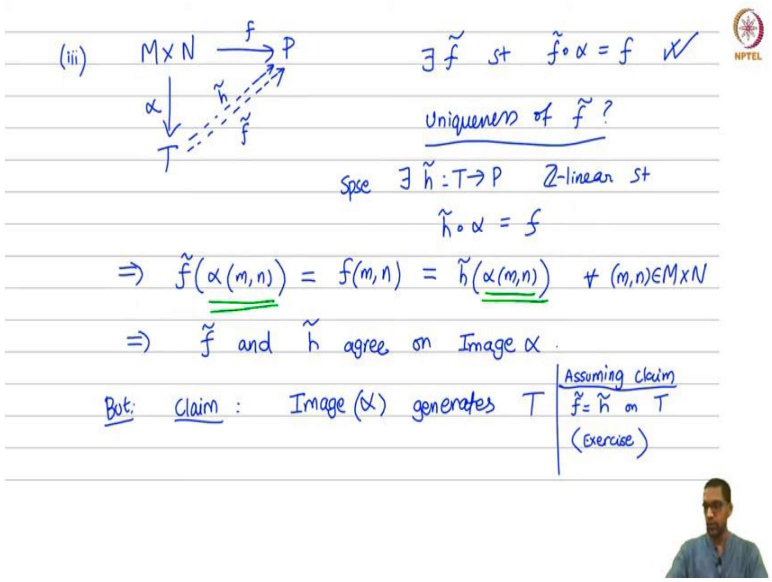
So, let us just show that this makes the overall diagram commute. So, what is the property we needed. So, I am going to find this as my map f tilde. So, now I need to check that, f tilde so need to check finally that f tilde composition, alpha is equal to f. So, that was the, that is the thing we need to do. So, at the moment let us get rid of all the intermediate steps. So, this is the big diagram we are looking at and the map alpha is just this overall map.

The composition of these two maps is what we are calling alpha and our claim is that, so this was called f tilde, this map, so f tilde composition alpha is f that was the universal property of the tensor product that we need to establish. Now why is this true? Well because each of the two smaller diagrams commute. So, how do we prove this? So, let us put all these back. So, now let us observe to prove this that, so first this upper triangle commutes, so proof.

Let us just do it, so first observe that g composition gamma is equal to f that is the upper triangle computing. Now the bottom triangle commutes, so which means g tilde composition pi is g. Now I will use I will substitute for g into the top equation. So, this means that g tilde composition pi composition gamma is equal to f, but pi composition gamma is exactly our alpha, that was our definition.

So, g tilde composition alpha is exactly f. So, that is exactly what we need to prove, I mean we are calling f tilde as g tilde. So, this is done. So, what has this done? It has shown that the pair. Therefore the pair T, alpha certainly has the first issue, I mean the first property, that there exists a map from T to the module P. But further we need to show that this map is unique. So, that is the last remaining bit of the definition.

(Refer Slide Time: 27:13)



So, we have shown that from M cross N alpha, given any pair P f there exists a unique, I mean there exists a map f tilde is okay. There exists f tilde such that f tilde composition alpha is f, that we have shown, but why is such an f tilde unique? That is the last bit.

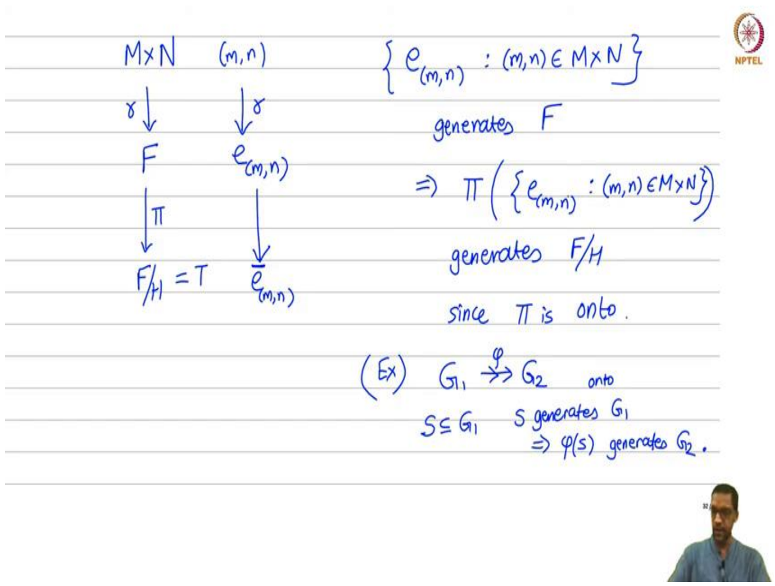
So, let us show uniqueness, so suppose I have another map. So, let us call this h tilde maybe. So, if I have another map Z-linear map, h tilde from T to P such that the diagram again commutes. So, suppose there exists a map h tilde from T to P, Z-linear map, also f, now observe what does this mean? This means that both f tilde and, so this is alpha of m n, is f of m n, but that is the same as h tilde of alpha of m n for all ordered pairs m n.

So, what this means is that f tilde and h tilde, so if you look at this f tilde and h tilde they agree on elements of the form alpha of m n. So, this means they are equal to each other, therefore f tilde and h tilde agree with each other, have the same values on the image of this map alpha. That is what this means, any element which is in the range of alpha or the image of alpha on such elements f tilde and h tilde take the same value because it is f of m n.

But the claim is that this image of alpha generates this group T, so that is the last claim we made that this image of this map alpha generates the abelian group T. So, once this claim is proved then it implies that f tilde and h tilde agree on all of T. So, observe that once claim is proved, so assuming the claim we are done, so maybe I will just write that on the side assuming claim, what do we obtain we can conclude that, if two homomorphisms agree on a generating set then they agree everywhere and then f tilde agrees with h tilde on all of T.

So, let me leave this as an exercise for you to prove that if two group homomorphisms agree on a generating set, then they agree on the entire group. So, let us prove finally the claim that is all remains, that the image of alpha generates the module the group T and that again is just from the definition.

(Refer Slide Time: 30:27)

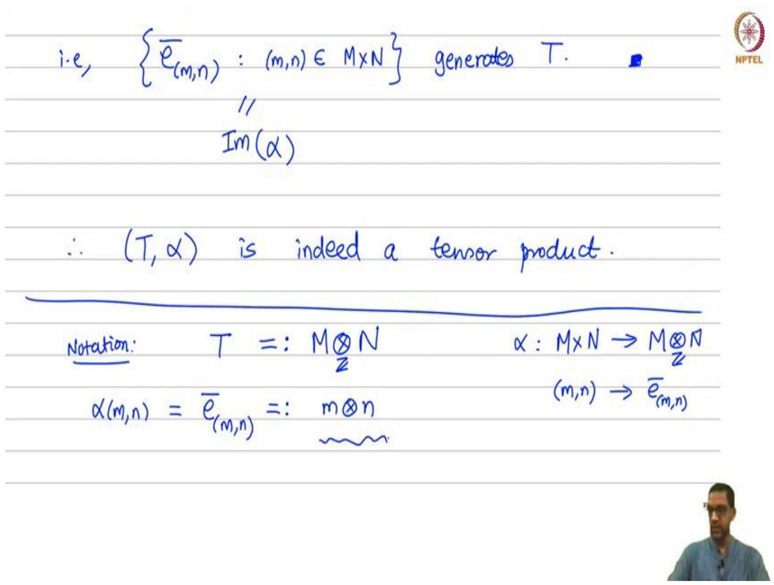


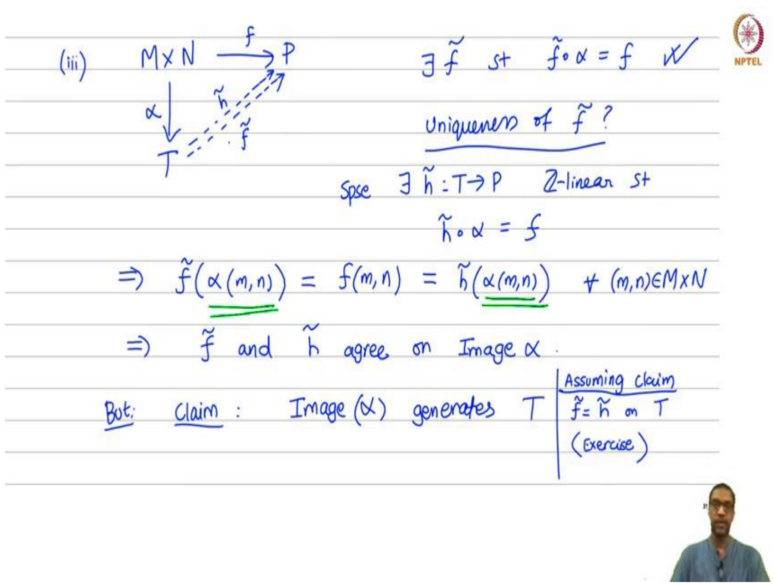
So, what was alpha? How was it defined? You first take this map gamma to the free group so each m, n goes to e sub m n. Now observe that the e sub m ns they generate F for sure, this we know e sub m ns, in other words the image of gamma, this is just the image of gamma that certainly generates the free module or the free abelian group, why is that? Because well, we already showed this, every element of F can be written as a linear combination with integer coefficients of the e m ns. So, if you have e m ns in some subgroup then that subgroup is the entire group F it is the entire module F.

Now, so they generate F and now this map pi here is a surjection. So, this is F mod H which is T. So, since pi is an onto map and this is a generating set it implies, if we apply pi to this generating set than that is again a generating set, this generates. F mod H since pi is an onto map. Again exercise this is just a completely general fact. If I have two groups and I have an onto map between them, which I will indicate by this double arrow so I have an onto homomorphism.

We will call it pi, then if a certain subset generates G1, so if S subset of G1 then S generates the group G1 meaning the smaller subgroup of G1 containing S is just G1 itself then its image generates G2. So, just a straight forward exercise using the definition of what it means to generate. So, when I have onto map and I have a collection of generators for the domain group then their images will generate the image. So, this is now, this goes to e m n bar. So, therefore the e, because pi is onto this means that the e m n bars

(Refer Slide Time: 33:05)



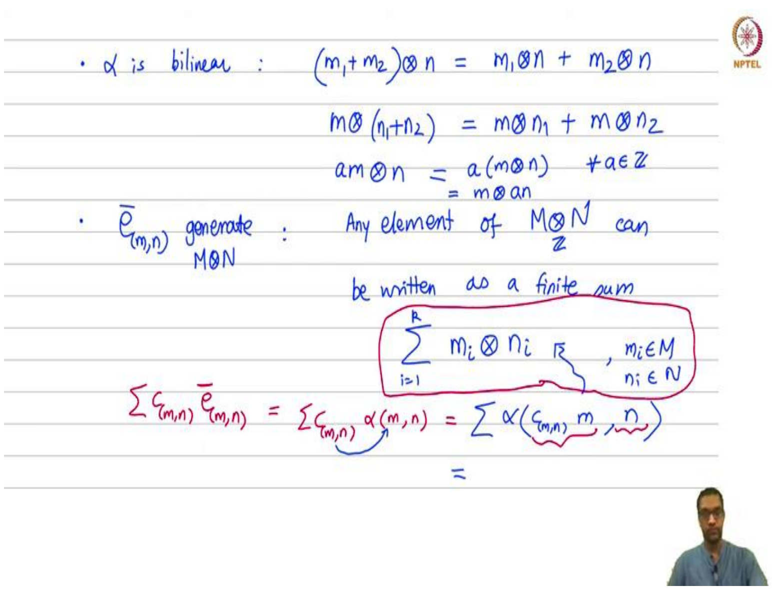


So, in other words i.e. what we are saying is that take these elements e m n bars inside T. They generate T. So, this collection of elements and therefore we are done. So, that is the last claim we needed to prove because this is exactly the image of alpha. So, therefore this shows that this map is unique. So, we have constructed the tensor product because we have shown that it has the universal property that there exists a map f tilde that we needed but not only that that map f tilde is unique. So, therefore conclusion finally, therefore final conclusion T, alpha is indeed a tensor product as claimed.

Now we have so, so couple of remarks on notation, which is that these, so we tend to write T as M tensor N, so this module T that we have constructed or in general any tensor product if you wish can be denoted as M tensor N over Z. The Z is just to say they are both Z-modules and we are thinking of the tensor product as Z-modules. Now this alpha also has a notation, so the alpha recall is a map from M cross N to T, so T which I am now calling as M tensor N, Now any, it is alpha. So, any m, n goes to what we call e m n bar.

Now this image we denote it by m, small m tensor small n, so this is again notation. So, this is notation for, so T is called M tensor N and this, maybe I should write this the other way around so this e m n bar, so this alpha of m, n is also denoted like this. You take little m tensor little n. So, that symbol will just denote the element alpha of m, n and, so what do we know finally? In terms of this new notation what we mean is that.

(Refer Slide Time: 35:43)



So, what we know is this, so alpha is bilinear, that in this new notation just says this if we take m1 plus m2 tensor n that is just m1 tensor n. So, recall this notation just says it is alpha of m1 n plus alpha of m2 n and similarly the other m tensor n1 plus n2, which means alpha applied to this pair, where ms are in M and ns are all in capital N and the other fact that we just said that the e m n bars generate the tensor product that was the other fact, what it means in this notation is the following.

So, recall this is just saying that any element of T can be written as a linear combination with integer coefficients of the e m n bars. So, this is therefore saying any element of M tensor N, this module just T can be written as follows. As well as a linear combination with integer coefficient c if you wish. Some finite sum as a finite of terms like this, so this is a finite sum it ranges over some finite collection, where m is are in M, n is are in N.

Why is this? Well, recall we know that we can certainly write it as a linear combination c m n in our earlier notation c m n e m n's are typical element of f and this is now when you apply the map pi, this gives you an element of T, but this I will rewrite in my, in this other notation so this is nothing but summation c m n alpha m n if you wish.

I mean there is a finite sum of certain ms and ns. So, what I am doing here is just calling the ms ns which appear in this summation as m i and n i but this coefficient c that I get, this constant here that I am just going to take into the, so recall alpha is bilinear which means that you also have this.

So, recall by the way when we said bilinear, so we also made this remark earlier that this also means that if I take a m tensor n, where a is an integer then this is just a times m tensor n for all a integers and this also the same as m tensor a n. Why is this? Well we prove this that bilinearity just means that you can, you know multiplication by a is like repeated addition those many times, so this last axiom just follows from the other earlier ones.

So this is, I can write this as alpha times, I can push the c m n this number, into the alpha because of bilinearity and now this is, this final expression is what I have written in this way. So, this is like saying. So, this is my, this is one of my m i, I mean this is my m i and this is my n i, and so you know what we have finally shown is that any typical element of m tensor n can always be written as a linear combination of the various e m ns and I can often in terms of you know when one wants to write down proofs and so on it is easier to just choose this more compact way of saying it instead of putting some constant c m n in front.