Algebra 2
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Lecture 61
Free abelian groups and quotient groups

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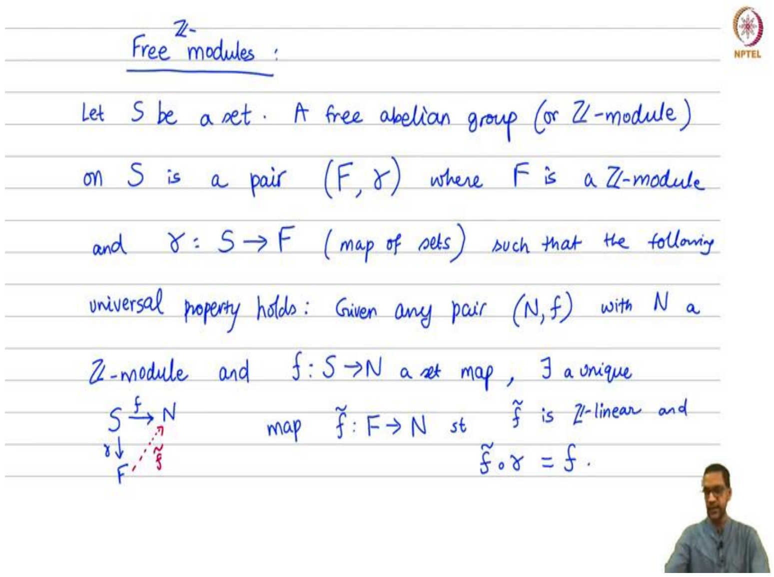

So, we have seen that the tensor product is a certain kind of a universal object but we still have not constructed the tensor product explicitly yet. So let us do that now, but before we do that let us look at two other examples of universal objects which will be required as intermediaries in the construction of the tensor products.

So, let me give you two more examples which you have seen in some form before. So, the first one is free modules or free modules of the integer free Z-modules or Free Abelian Groups. So what is the free abelian group on a set. So given a set, so let me define it like this a free abelian group or Z-module on S is again a pair F comma gamma where F is a Z-module and gamma is a map from S to F just a map of sets, just a function.

With the additional property such that the following universal property holds and what is the property? Given any pair like this, given any pair N comma small f with N, Z-module and f a set map and as before we will draw the same sort of diagram that we drew for the tensor product definition itself, so S to F I have gamma. Now I am going to take an arbitrary pair S to N, there is an f then given any such pair what the universal property asserts is that there is a map f tilde from F to N which is a homomorphism of abelian groups.

So, given any such pair like this there exists a unique homomorphism. There is a unique map f tilde from F to N such that two properties, f tilde is a homomorphism of groups. In other words it is a Z-linear map and property two, the diagram commutes.

So, this is the definition if you wish of a free abelian group, which says that there should be a map firstly from the set to the group f that map is gamma such that given any other such map from the set to any other group that map factors through f, which is exactly how we express this fact that there is a unique map from F to N which makes the diagram commute. This may look like a slightly strange definition if you have seen other definitions of free groups before, but they are all the same sort of definitions.

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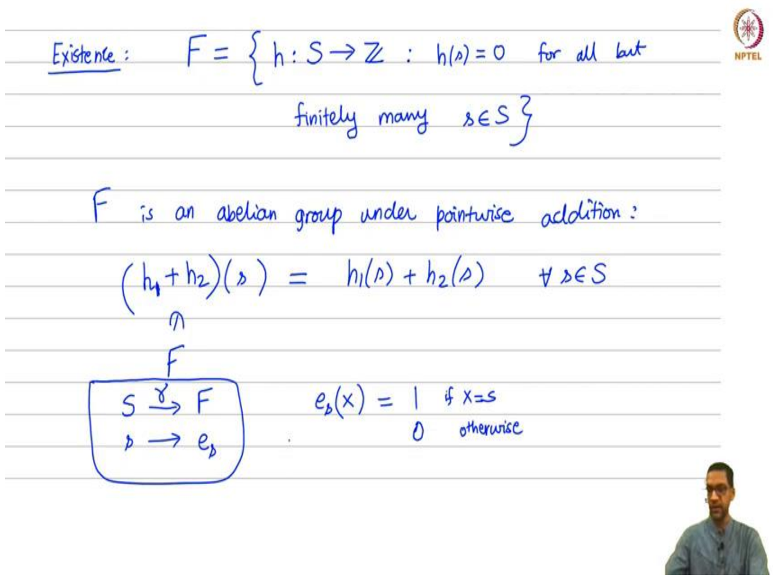

So, first let us show using this definition why the free group exists and here the set S can be finite or infinite it does not matter. So, how do we prove existence of the free group on a set S.

So, let us construct the group. So, how shall we construct it? Well the set and the group F as a set is easy to describe. Let us take all functions from the given set S to the integers. It is just all functions, but well I do not want h to take non-zero values for infinitely in the S so I will say I will put the conditions called finite support that h of S should be 0 for all but finitely many values of S.

Outside this finite set of finite subset of S for all the other values of S, h of S is 0. So, it is functions of finite support from the given set S to the integers. Now why is this group? So, firstly observe that F is an abelian group, so the claim is that F is an abelian group or a Z-module under point wise addition.

In other words, how do you add two elements. So, if I give you h 1 plus h 2 two element of F you add them like this, the value on S is just h1 s plus h2 s. This is for all s in S. So, this is a new function. Now the point is if h1 and h2 are 0 outside a finite set then, so is there point wise sum, why is that? Because look at the set of points on which h1 takes a non-zero value, take the finitely many points on which h2 takes an non-zero value, then h1 plus h2 can only take a non-zero value on the union of those two sets.

Outside the union it is definitely 0 and the union is definitely finite, because both, you take the union of two finite sets, it is finite again. So, the key point is that h1 plus h2 if you make this definition again it is a finitely non-zero function or a function of finite support.

Now let us understand this abelian group a little bit more. So, there are some special functions on this, so let me just call this for each S. So I have the following, so look at this, associated to each s there exists a map. I should say, let me construct this map from S to F first. So, to each S I have to associate a certain element of F, certain function and this is the function defined like this e s the function on a given element of s it takes the value 1 if x equals s and 0 otherwise. So, this is the function, so sometimes called the indicator function of that point. It just takes the value 1 on just that single point S.

So, this is of course an element of F and now I have defined a map from S to F and this is going to be my map gamma. Now observe that if I define these functions e s x in this manner then these are like the coordinate functions if you wish.

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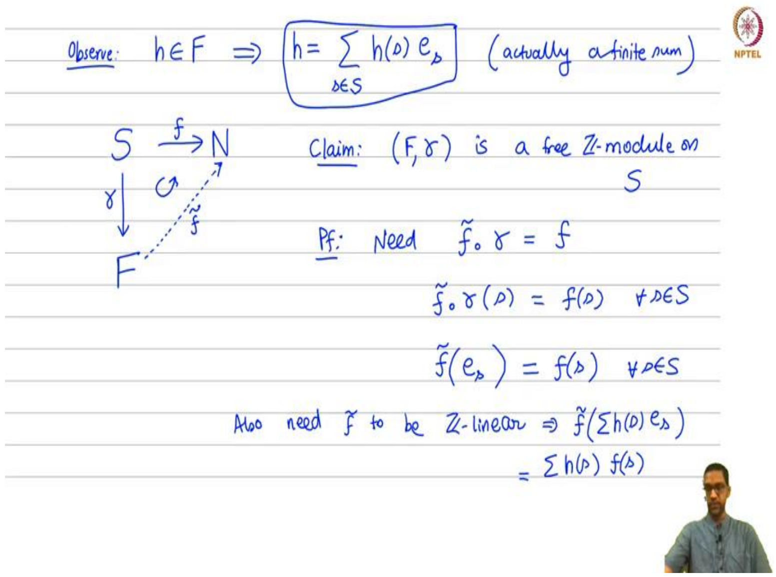

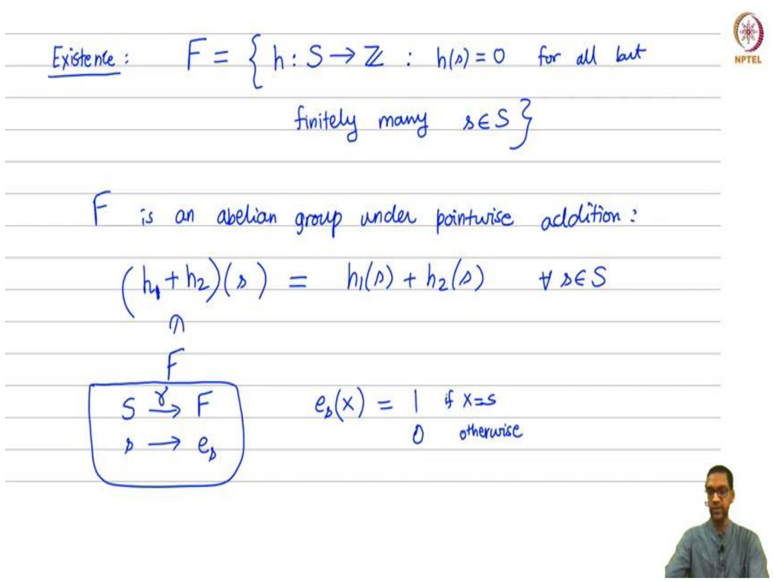

So, observe that every element of F I mean this is sort of from some sort of basis over Z. So given any element of F implies h can be written as follows. It’s h of e sub s. This is s running over S but really this is only a finite sum. This is actually only a finite sum because h is a function of finite support.

So, anything can be written as a linear combination. Any function can be written as a linear combination of the e s S. So, what all have you done so far? We had a set S. Now we have defined abelian group F and we have defined a map, set map from S to F. Now let us show that this in fact has the universal property that we need.

So, F gamma is a free Z-module on S. In other words, we need to show given any N and any function f I need to show that I can construct a unique, there exists a unique group homomorphism from F to N which makes this diagram commute. So, let us prove this. So, what does the given condition satisfy? I mean what is the commuting of this diagram. So, sometimes we put this arrow to say that that diagram commutes. So what we need, we need the following property that f tilde composition gamma should be f. In other words, let us evaluate both sides on S for all elements of s of S and what is this? This is gamma of S we just defined.

If you go back and look gamma was defined to be the map which takes each element S to the corresponding indicator function e s. So, this f tilde takes e s to the value of f at s. So, this is what f tilde must do. So, what does the imply? Well that tells you so if at all you can find such an f tilde then that f tilde has to satisfy this property.

And, so what is the other possibility. So, I mean what does that automatically imply and we also want f tilde to be a group homomorphism. So, we need this property also need f tilde to be a group homomorphism meaning it is a Z-linear map. So, what does that mean? When f tilde acts on a linear combination, so look at the linear combination that we wrote about, the answer will just turn out to be, I mean you can pull the scalars h of s outside.

And f tilde of e s is just f of s, so in some sense what this tells you is, if suppose such an f tilde exists then we actually know what its formula is. It has to be given by this formula there is no other way out. So, it is definitely unique. The thing that remains to be proved is that this formula actually defines a group homomorphism. So, this is just sort of the initial heuristic motivation if you wish. So, now let us go ahead and define.

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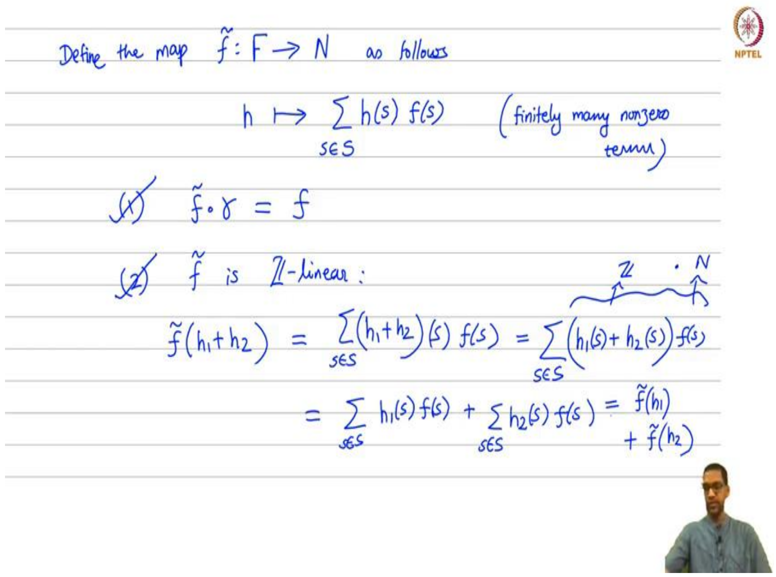

So, let us define the map f tilde from F to N as follows. Given any element of F I will map it to summation h of s f of s. This is s belonging to S and again observe that this sum has only finitely many non-zero terms. So, this is only finitely many non-zero terms, because the element h has only finite support.

So, that is my definition. So the claim is that this map is, so observe the following fact that f tilde, this particular map certainly makes the diagram commute. In fact that is how we deduced what this formula should be, the key property to prove is that f tilde is a group homomorphism. In other words it is Z-linear.

So what does that mean? We need to show that if we take f tilde and take a sum of two guys, h1 plus h2 we may see what the answer becomes, so it is h1 plus h2 evaluated at s times of f of s but h1 plus h2 is just the point wise sum. h1 s plus h2 s multiplied by f of s and this is now summation h1 s f s plus h 2 s f s, because well now here I am using the fact that all of these are elements in the Z-module N. So, observe this is an element, I mean this is a scalar, this is an element of Z. This part is an element of N f of s.

So, what we are doing is now N is a Z-module, so when a scalar multiply a sum of two scalars you have distributivity. So, I am using the fact that N is a Z-module and so I have split it like this and now observe this is exactly f tilde of h1 that is the first sum plus f tilde of h2. In other words, we have shown that f tilde is Z-linear. So, what does that mean, it shows that, so it has shown the universal properties or claim has established. So, we have constructed the free module.

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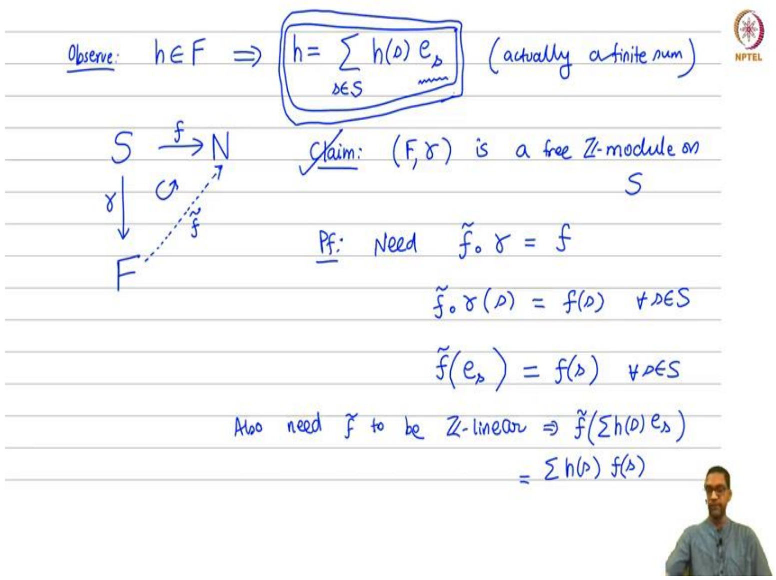

So, the key point really is this, so if you remember what is a free module on the set S. It (())(15:05) elements of the set S form a basis for this module, basis over Z in some sense and here that basis is really like the e s s if you think of the function e s as being like a proxy for the element S. So, we have constructed the free module, again because it is a universal object we have the same principle as before. It becomes an initial object in the category suitably defined and so on. Like we discussed for tensor products.

But in more concrete terms it says that not only does the free abelian group exists, our existence proof shows it exists but then it is also unique, up to a unique isomorphism. In other words if I take another candidate F prime, gamma prime then there will be an isomorphism from F to F prime which makes this diagram commute and further that isomorphism is unique.

So, that is exactly the same discussion that we had earlier for the tensor products in the previous video and that also follows from the general discussion that you have seen because this is an initial object in a category. The initial object always has a unique map. I mean any two initial objects in a category are, there is a unique isomorphism from one to the other. So, now we have constructed one of these.

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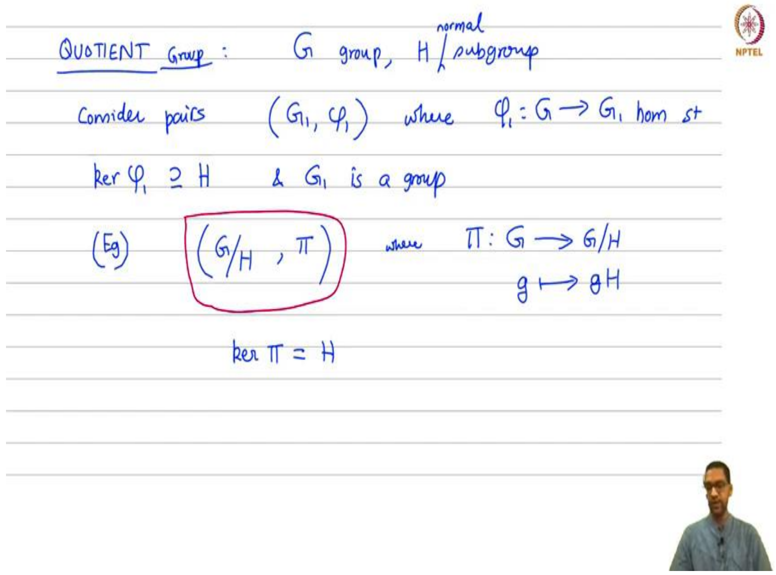

Now the second thing that I wanted to talk about in the terminology or context of universal properties is the notion of a quotient, the quotient group really. So this is, in fact this is even for non-abelian groups really, so if G is a group and H, suppose I fix these. H is a subgroup. Now what I am interested in is if you wish the category whose objects are going to be pairs, consider pairs, pairs what, of G 1, phi 1 where, what is phi 1? It is a map from G to G1 such that the kernel of this map contains the fixed subgroup H, so we have fixed G and H here and I am looking at pairs and G is a group I should say, G1 is a group.

Phi is a group homomorphism. This is a homomorphism such that. Now consider all such pairs, for example so there is one obvious such pair which is you can take G1 to be, I mean the pair to be I should say normal subgroup. So, suppose H is a normal subgroup of this group G. Now let us look at one possibility, let us look at the quotient group G mod H and what is this map that I want to consider pi.

So, consider the following pair for G1 I will take G mod H and for this map phi 1, I will take the projection map pi, where what is pi? Pi is the obvious suggestion G to G mod H which takes every group element to its coset. This is the natural map from G to the quotient. Now this is of course one such example in fact in this case the kernel of this map pi is exactly H.

So, it surely contains H. Now the point that I want to make in this setting is that this particular pair is actually also a universal object. This is also universal in the following sense. If you consider all such pairs G1 phi 1 then the following is true.

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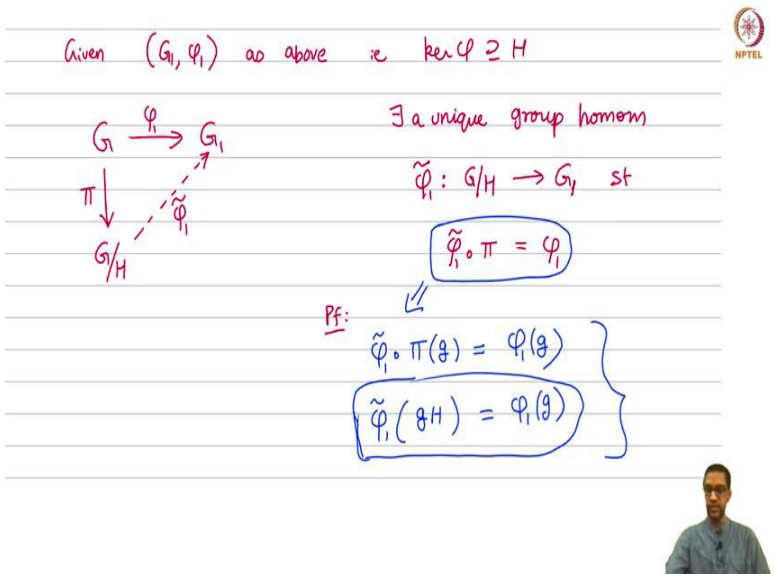

So, given G1 phi 1 as above, meaning its kernel contains G to G1 we have the map phi 1 and the kernel i.e. kernel of this map contains H then here is what I can say. So, here are two possible pairs but if the kernel of phi contains H then there is always a map like this. So, this is phi 1 tilde then there exists a unique group homomorphism phi 1 tilde from G mod H to G1 such that the diagram commutes.

Such that phi 1 tilde composition pi is phi 1. So, this (())(20:19). So, let us prove it. It is rather straightforward. So, what do we want, we sort of know the commuting property the diagram should commute. This tells us something. It says that if I apply phi 1 tilde to pi of g, the answer should be phi 1 of g.

In other words, phi 1 tilde acting on, what is pi of g? We just said it is the coset gH and of course this what should be the case if the diagram has to commute. So, we sort of know how to define this map phi 1 tilde. There is really only one choice. You must map the coset gH to the value phi 1 of G. So, this is like I said, like we did in the previous case. This is the sort of the heuristic motivation. It tells you what to do. And now you just have to check that if you actually do that then it is well defined group homomorphism. So, let us define.

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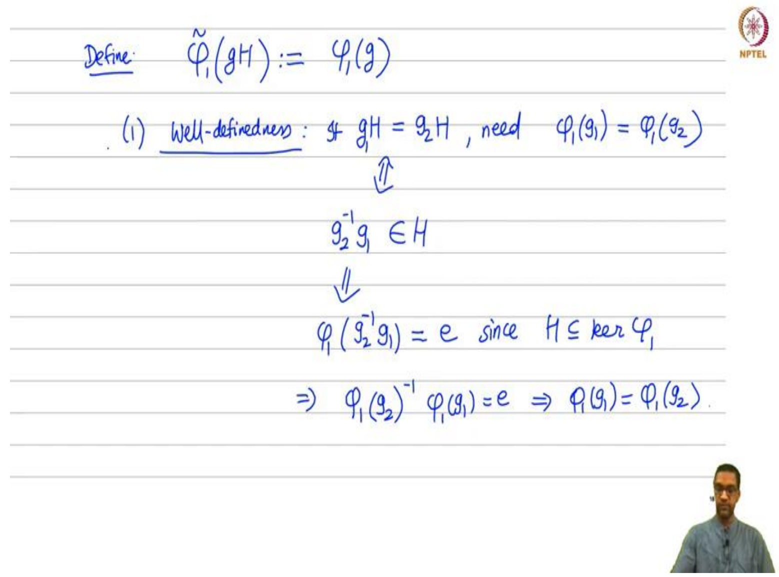

Now go ahead and define phi 1 tilde as follows on the coset gH you define it to be phi 1 of g. So this is going to be a definition. Now the first thing whenever we define a map like this on coset is to ensure that this is well defined. That it does not depend on the representative that we have chosen for the coset.

So, you always have to check well definedness. Now what does that mean? It says that if the same coset has another representative so suppose g1 H and g2 H are both, g1 and g2 are two representatives for the same coset, then let us see what would have happened. If this is true you need to show that whether you use g1 for your definition or g2 for your definition the answers are the same.

So, phi 1 of g1 should be the same as phi 1 of g2. This is what I need to prove, but let us see what do we know. g1 H equals g2 H is the same as saying in fact that g2 inverse g1 belongs to H but remember H is contained in the kernel so in particular if I evaluate, since H is contained in the kernel.

So, I have used the property that is given to me and now I use the fact that phi 1 is a homomorphism, so it is phi 1 of g 2 inverse, phi 1 of g1 is the identity. So, that is the same as saying phi 1 of g 1 equals phi 1 of g2. So, therefore it is well defined. So, I have checked the well definedness. I just need to check that it is a group homomorphism and that is easy. So, let us check.

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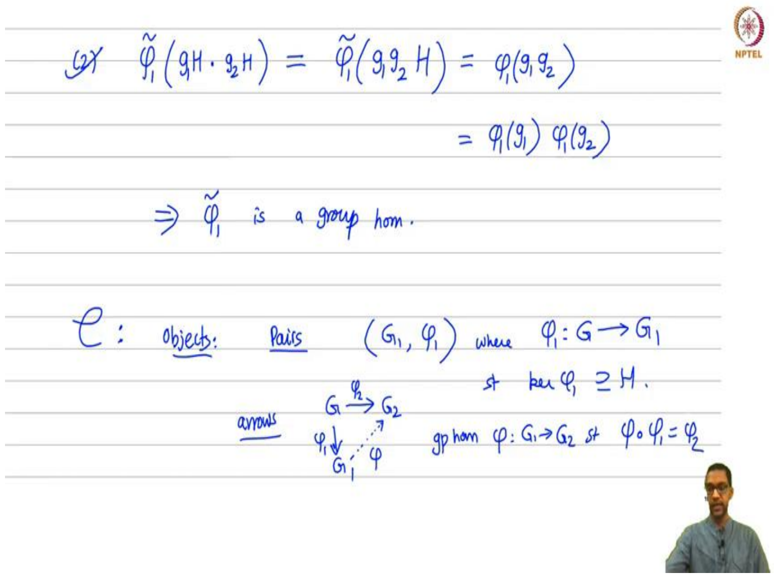

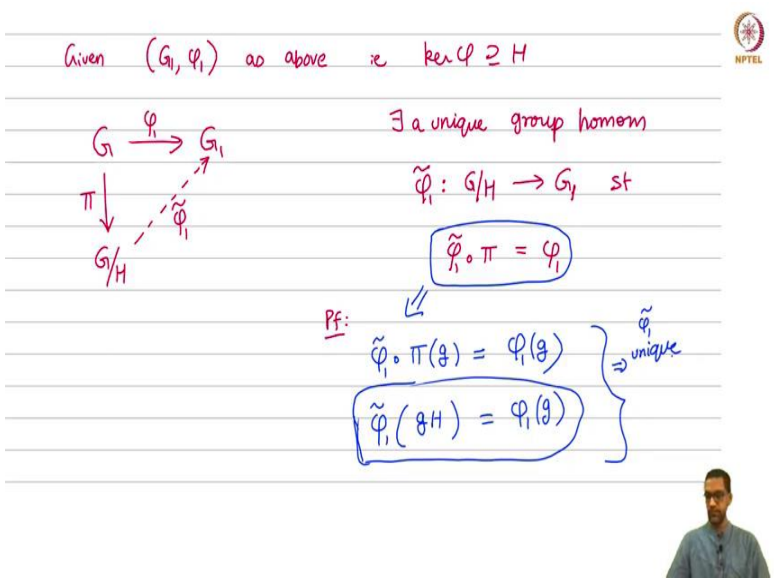

Phi 1 tilde, how do I check it is a group homomorphism? I take a product of two cosets. Well the product of two cosets if you remember is just the same as the coset of g1 g2, H is a normal subgroup and this is how you define the product of two cosets and now this by definition is just phi 1 of g1 g2, but phi 1 was a homomorphism to start with, so this phi 1 g1 times phi 1 g2.

So, this means that phi 1 tilde is a group homomorphism. So, we have checked both the properties and uniqueness is of course true because of the way we, I mean this was the observation we made in the beginning. This argument implies that phi 1 tilde is unique, because the definition is forced upon us.

So, we have in fact shown all three parts. So, this is again you can think of this in terms of any initial object of a category and so on. You will have to define your category as follows. The objects are now pairs of, well what are they pairs of? Group and homomorphism. So, what is this? Phi 1 is now a homomorphism from the fixed given group G to the group G1 such that its kernel contains the group H.

So, this is going to be, these are the objects and what are the morphisms or what are the arrows in this category? Well you have one such pair G1 to G2 and another, let put G1 here and have G2 here then, what is an arrow oe morphism? Well it is a map, it is a homomorphism, let us call it phi, so an arrow by definition now is a group homomorphism phi from G1 to G2, which makes the diagram commute. So this your, this is an arrow in this category. So, once you define your objects and arrows in this particular way.

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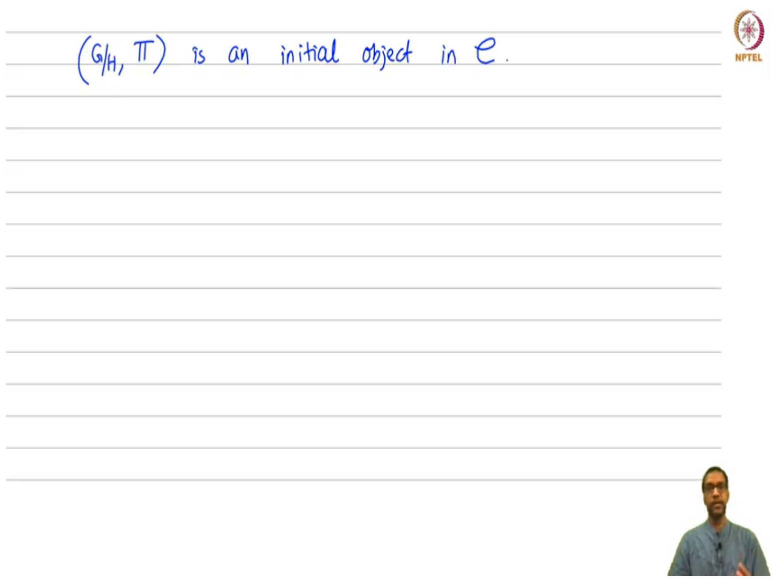

Then observe again that the quotient that we are taking about G H, pi is an initial object in this category C. So, all these universal properties can be phrased in terms of initial or final objects in appropriate categories. Now we will use this next time to actually construct the tensor product of modules.