Algebra - II
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Lecture 6
The Field of Constructible Numbers

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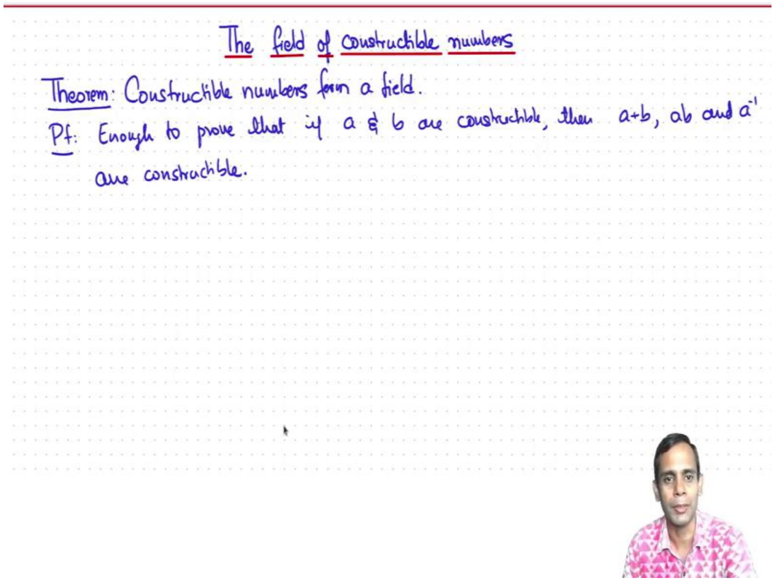

In this lecture I am going to show you that constructible numbers form of field. So to show this it is sufficient to prove that if I have two constructible numbers then a plus b, ab and a inverse 1 over a are constructible. So to do this we will make use of a certain very standard construction with straightedge and compass which is the construction of parallel lines, of course you have seen all this before but let us just do this carefully once using GeoGebra.

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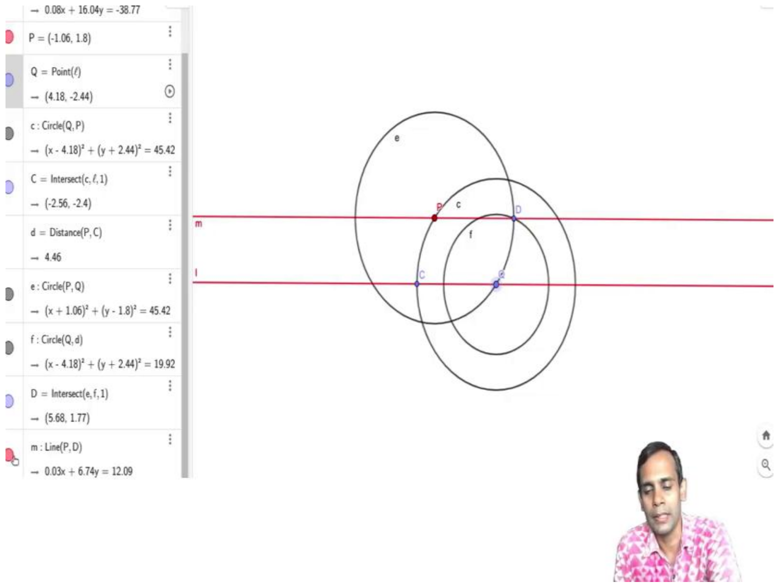

So the construction is the following you are given a line l here and a point P and the task is to construct a line that passes through P and is parallel to l. Now the first step in such a construction is to somehow find a point Q that lies on the line l. So to do this usually you are given so remember when we talk about what a constructible point is, we always started with given two points which we call O and A.

So if the lines segments O and A is not parallel to the line l then you can just extend it to intersect l at some point and that would be, that could be a point Q. If O A is parallel to l then you could construct a line that is perpendicular to O A something we did last time and then that line would intersect the line l in a point which we would call Q. So in any case you would be able to somehow find a constructible point on Q.

So somehow construct point on Q and then we can start our construction. So what you do is the first step you draw a circle with center at Q and passing through P. And now you mark as C the intersection of this circle with the line l. And now let d be the distance between the point P and C, so this distance we are calling d. And let e be a circle passing through P with center Q.

So I have the second circle now with center P at passing through Q and the idea is I will measure of an arc on this circle e centered at Q with the same arc length as the arc of this circle C passing through P and capital C so what I will do is I will draw a circle f with center at Q and radius equal to d. d is the distance between P and capital C and then I take capital D to be a the intersection of a the circle e with the circle f.

And now let us define a line m that passes through P and D then l is a line that is parallel to, m is the line that is parallel to l and passes through P. So with this construction you see that is possible given a point a line and a point not on it to draw another line that is parallel through to the original line and passing through the point the given point.

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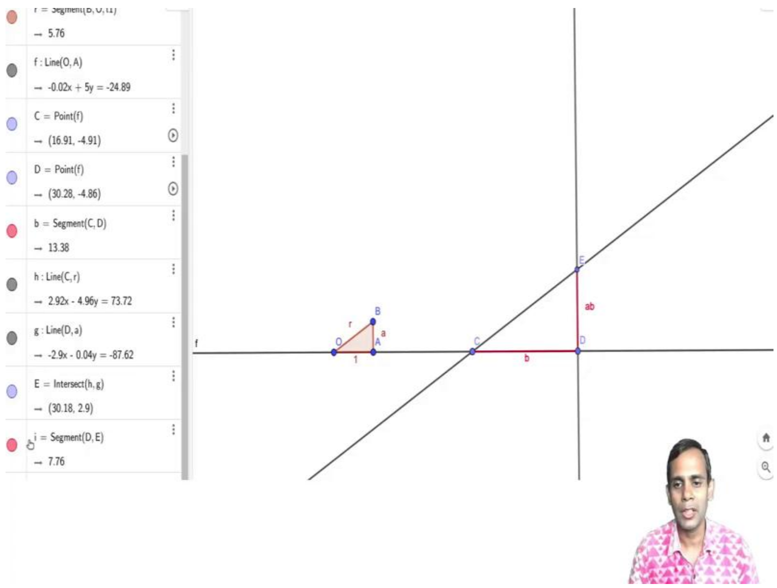

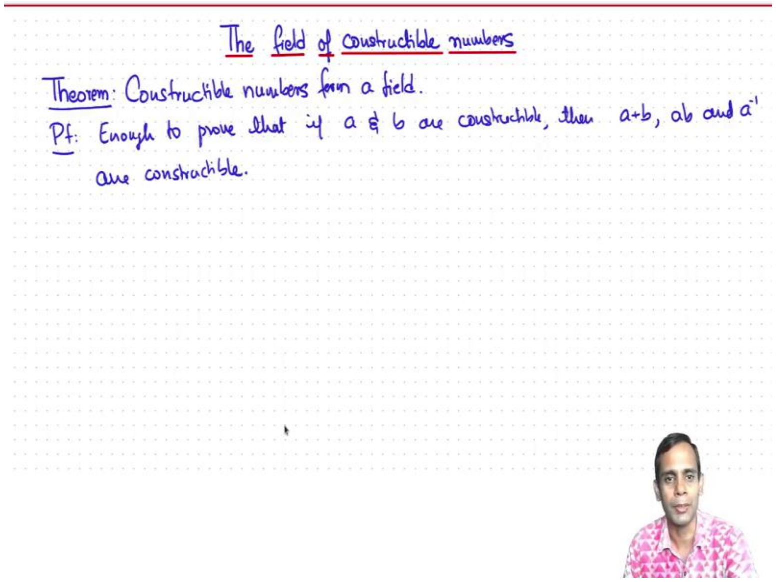

So now we will use this construction of parallel lines to construct products a b and a inverse, it is quite easy once a and b are construct, you know if you are have points distance a the distance a between them and another pair of points with distance P between them, then it is very easy to construct two points with distance a plus b between them you just take, along a line you first take an arc of length mark of length a and then you mark of length b using your compass and then totally you would have marked of a plus b.

So the interesting questions are given a and b can I construct a b and given can I construct a inverse? So we look at those constructions. So now let us assume I am given a two lengths a and b. So what I am going to do at first is I am going to construct right angle triangle like you see here on the left hand side with base 1 and that base could be the original points given O and A and I am going to erect the perpendicular at A, at capital A of height little a.

I am just going to mark of little a along this perpendicular line. So I have this right angled triangle O A B and the idea is to construct another right angled triangle whose base is b. So how do you do that well you mark off the length b again somewhere along this line passing through O and A. So I have mark them off here the points the distance between the points C and D is little b. And then you draw a line through C that is perpendicular to this line through O and B that line segment we called r.

So we have drawing something that is parallel to r and passing through C and then you also draw a line that is parallel to A B but passes through D. So what we have in effect is a triangle that is so if I call this new point E the intersection of these two lines then we have a triangle C D E that is similar to the triangle O A B. And so if you ask what is the length of this line segment E by D then in the length of line segment E by D divided by B is the same as the line segment A by B divided by 1. So that means that the lines segment E D has length a times b and so given a and b is constructed the product ab.

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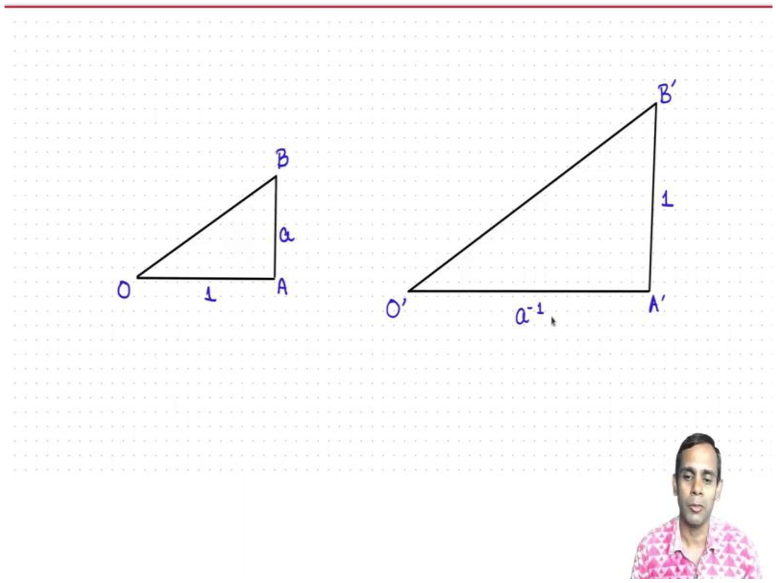

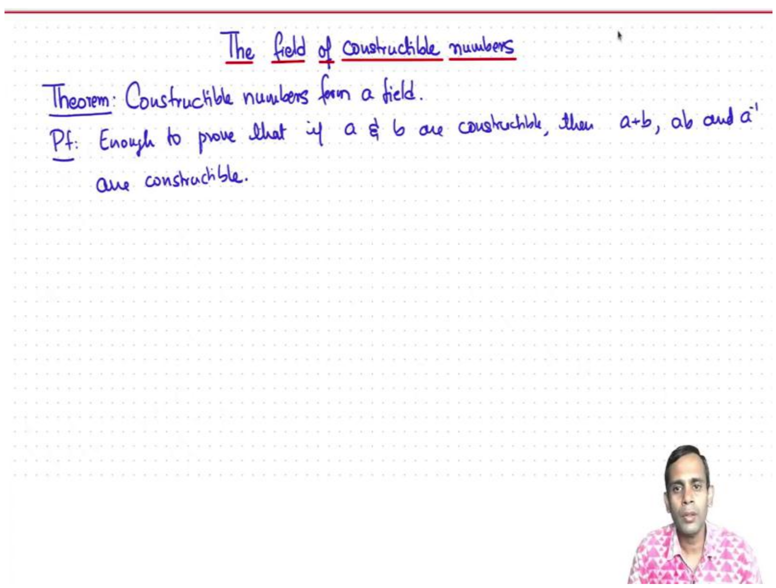

So now it remains to show that if I can construct a then I could construct a inverse. So the idea here is very similar to the construction a b and so slight variant again using similar right triangles and I will just show you the general idea and give you to fill in the details. So the point is you start off with a triangle O A B where the vertical edge is of length a and then you construct a triangle that is similar to it.

Let us call it O prime A prime B prime where the vertical edge is of length 1. Now if this vertical, now the ratio A over 1 is the same as the ratio of 1 over this edge O prime A prime. So that is means that this O prime A prime is of length a inverse and so if you can construct A then you can also construct a inverse using straightedge and compass and this shows that all constructible numbers indeed form a feet.