Algebra-II
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Lecture 59
Tensor products of Z-modules

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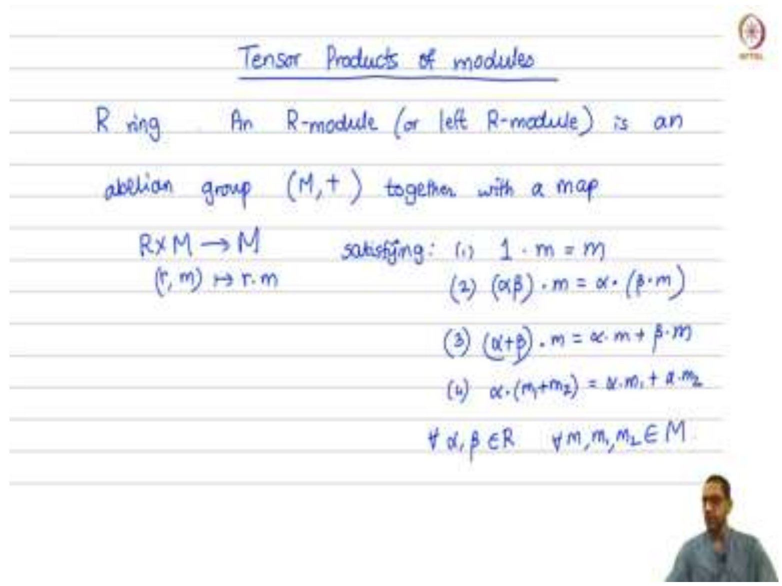

We will start taking about an important notion about modules, the notion of tensor products. So, recall what a module was. So, we spoke about this in Algebra-I so you have a ring R. Now an R-module or a left R-module if you wish is an abelian group M, it is called the operation as plus together with the notion of scalar multiplication, so together with a map.

So, there is a scalar multiplication from R cross M to M. which is denoted r comma m goes to r dot m. So, this is scalar multiplication by R satisfying the following axioms. Satisfying a list of axioms so number 1 the identity, multiplicative identity of ring. When a scalar multiplies any element you get back the same element, alpha beta, similarly alpha plus beta, acting on m gives you alpha dot m plus beta dot m and fourthly alpha acting on m 1 plus m 2 gives you.

And now this is supposed to hold for all scalars alpha beta from the ring R and for all m, m 1, m 2 elements of the module M. So these are the main axioms of scalar multiplication for example like in vector spaces and so on. So, module is sort of like a vector space but over a ring instead of over a field.

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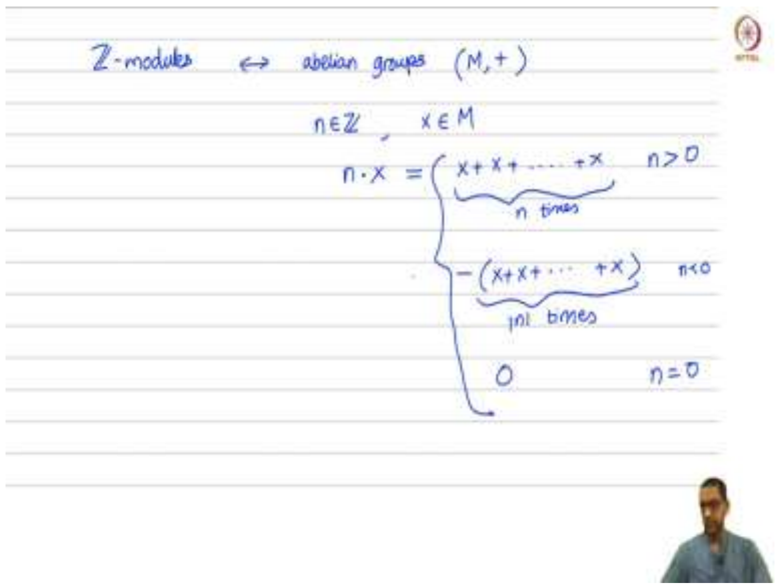

Now, the key examples that we will initially consider at least are modules over the ring of integers Z-modules and recall again that these are the same as just abelian groups. So, Z-module structure it does not add anything extra. It is just the abelian group structure on M itself. That is all you get. So, what is the action, so if I give you n in Z and I give you an element m in M, we will call it x in M, then how does one define n multiplying x, the scalar multiplication?

Well the answer is it is just x plus x plus x n times so you just add x with itself n times. So, this is n times, of course, you can only do this if n is a positive integer and if n is negative you just declare it to be minus of x plus x plus x, well n is a negative number here so I should take modulars of n times. Or I would say it is 0 if n equals 0. So, these are the three, I mean think of it as three cases. So, that is the definition of scalar multiplication by the integers recall and this is well.

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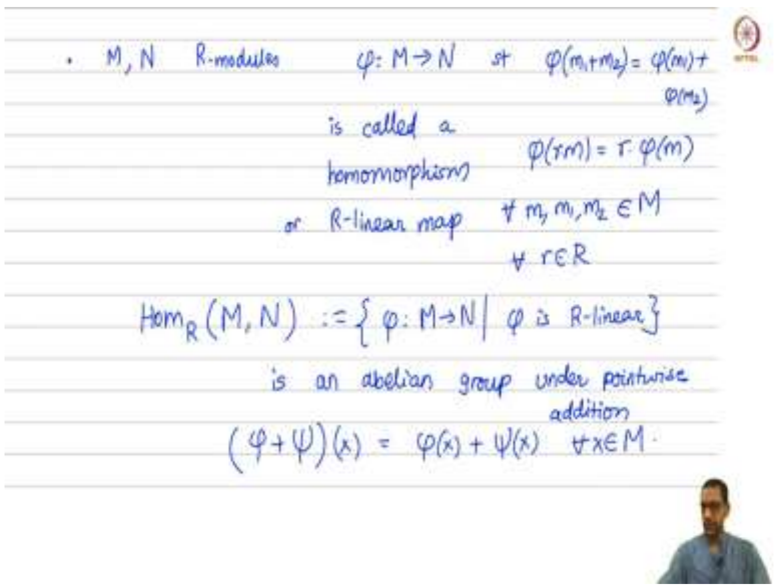

So, Z-modules are the same as just abelian groups themselves. Now the other important notion is that of R-module homomorphisms so recall that if I have two R-modules M and N R-modules then a homomorphism is a map phi from M to N such that phi of m 1 plus m 2 and phi of rm equals r acting on a.m. is just r acting on phi m for all m, m 1. So such a map is called a homomorphism, module homomorphism or R-linear map.

And the set of all such maps are linear maps. We denote by Hom R M N, okay this is just is a set of all phi from M to N such that phi is a homomorphism and we have seen earlier in Algebra-I that this is actually an abelian group under point wise addition by which we mean if you add, so I should tell you how to add to homomorphisms and that is the operational you just perform addition point by point. This is the definition. And it is easy to check that it is an abelian group under this definition.

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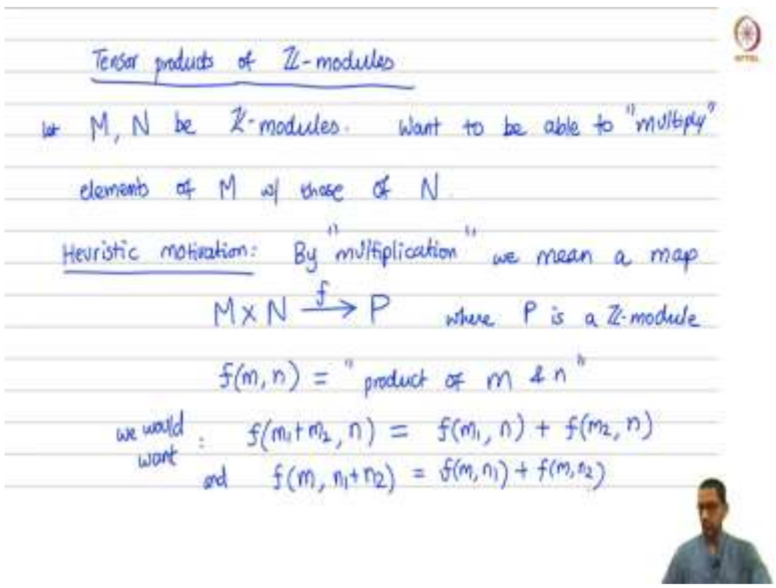

Now, let us get down to tensor products. So, what is this idea of tensor products, so let us start with some heuristic motivation of what tensor products are supposed to be and for a start I will talk about how to construct tensor products of Z-modules or abelian groups. So, suppose I am given two abelian groups M and N. Now what we need to do, well we already know how to, well given elements of M we know how to add them together given elements of N we know how to add them together. What we want is some notion of multiplication.

So, we want to really be able to want to be able to multiply that is where, so I will put multiply in quotes. Multiply elements of M with those of N. I will now try and construct some notion of multiplication. So, observer for example that is I take M equals N then this is sort of how you, this is where the ring structure comes up.

You start with an abelian group structure and then you try to introduce some multiplication in it, and that is what a ring is really, right? Provided it satisfies the axioms. So, now let me just give you some heuristic motivation. So, what is a multiplication. First we have to try and understand what that means, by multiplication we mean a map, okay so by multiplication now of elements of M with those of N we mean a map.

Now what is this map it takes as input an element of M and element of N and the output is an element of some other third abelian group, some other Z-module P. So, when we say multiplication this is roughly what we want to do. So, what does this do? So, where P is some Z-module and, so what is it that we would want to do, f of m and n is what we want to think of as the product of m and n. The product now lives in the space P.

Now what do you want, what sort of axioms would you want this map to satisfy? Well the natural axioms that you would expect of multiplication, which is that multiplication distributes over addition that is the key axiom. So, we would want this such a, if you have really constructed a nice candidate multiplication map then you would want it to satisfy this sort of distributive property. So, imagine this one, it is like m 1 plus m 2 multiplied with n.

You would want the answer to be m 1 multiplied with n plus m 2 multiplied with n. And you would want the same thing to hold in the other component m comma n1 plus n2 equals f of m m 1 plus f of m m 2, and now this is true for all m's and n's in the appropriate spaces.

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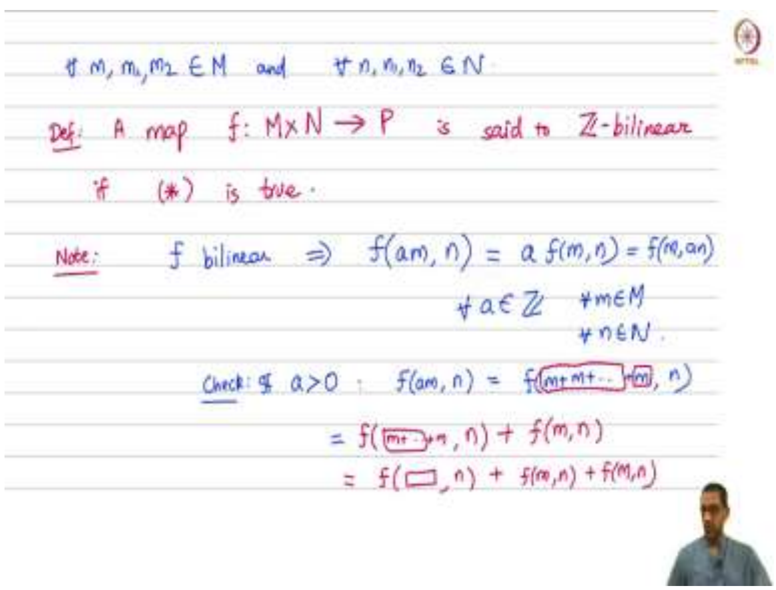

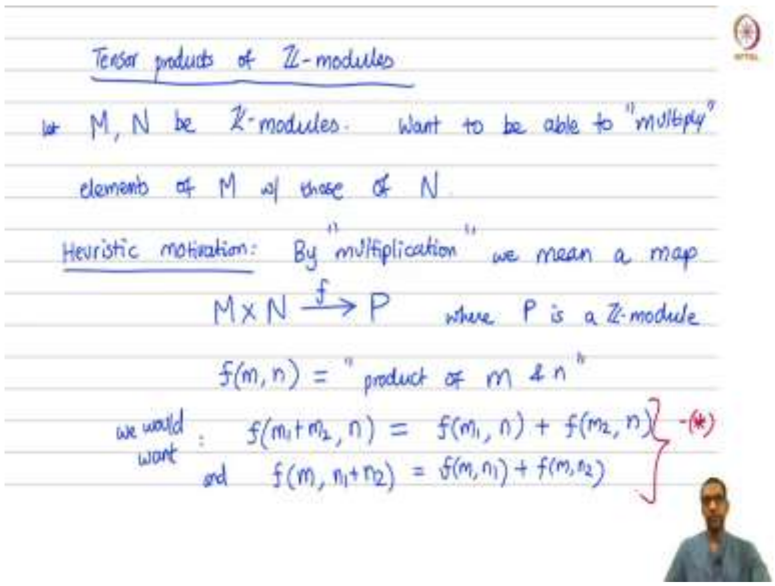

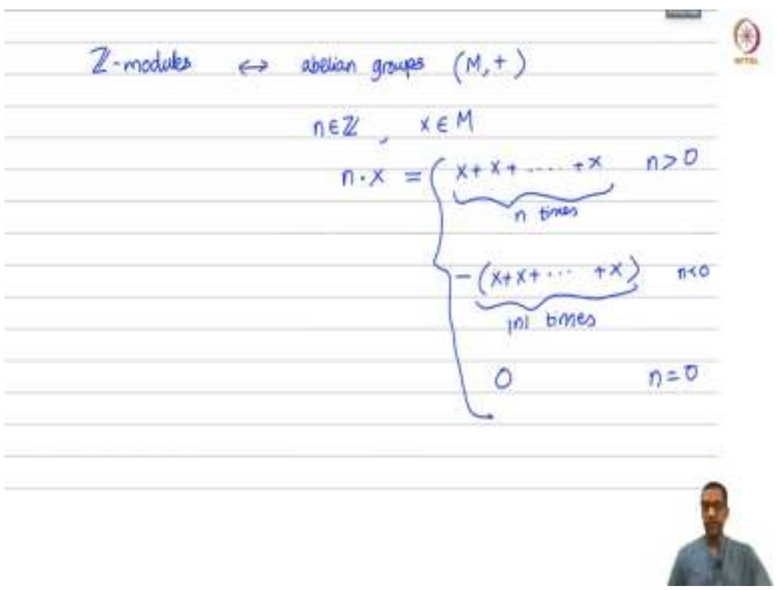

So, this is a notion of a multiplication to you. Okay, now let us make this a bit more precise so any such map which satisfies these two axioms. Let us, it could be the name star so a map. So, here is the definition. A map f between the, from the cross product M cross N to a module P is said to be bilinear or in this case everything is a Z-module, you say this is Z-bilinear if the equations star is true.

And star remember is just these two equations here that f of m 1 plus m 2 comma n is f m 1 n plus f m 2 comma n and then a similar property in the other component. So, this is sort of saying linear, the first component, the second equation says it is linear, the second component. So, such a map it so to be bilinear.

Now, observe also that so little fact, observe that a bilinear map also satisfies the other, well an additional property if you think of it. If it is bilinear also means that f of, well if I multiply, so let me say am comma n is as same as a times f of m n. Now what is a here, so recall everything, this is Z-module, M is a Z-module, N is a Z-module, P is a Z-module so I claim that all these three quantities are equal. For all a in Z, for all m in M and n in N.

Now why is this true? Well just by the way the Z-module action was defined. So, let us recall how this was a reaction. When you multiply some element of the module by an integer then you just have to add it those many times. So, in this case the integer is a so for example let us check one of these guys, maybe the first equality.

So, I mean I have to take the various cases so suppose, let me just check for one case so if a is positive then this first quantity f of am is just I have to write this m many times comma n, but observe by linearity also means that it is completed distributed in the sense that, so I can think of this everything except the last m as one single object, one single box. This has another element of m and use the distributivity or bilinearity in the first, linearity in the first axiom.

So, what that means is I get f of all these other m's comma n plus f of the last m comma n. now I repeat the same process. So, this has now got one fever m now I will put, bunch the remaining m's together, keep the last one separate and then again use the bilinearity axiom. So, again the number of m's will decrease by one but then here I get two times the same thing. I get the same term twice and so on.

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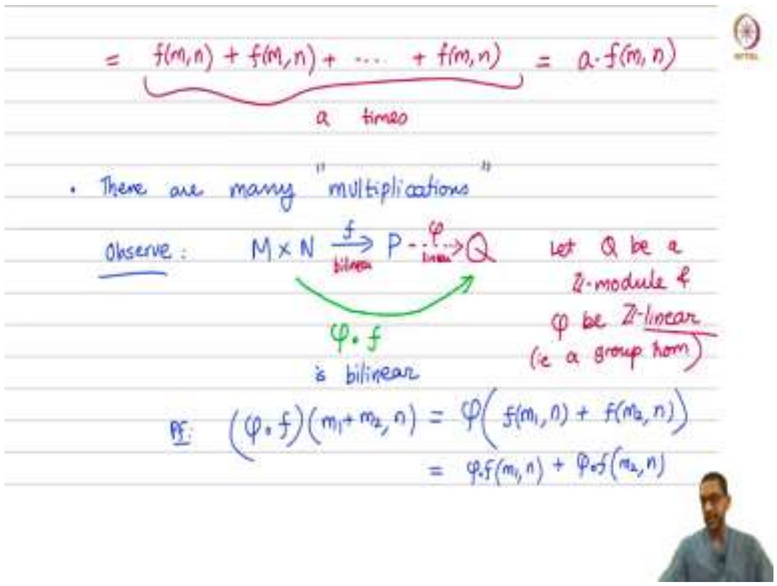

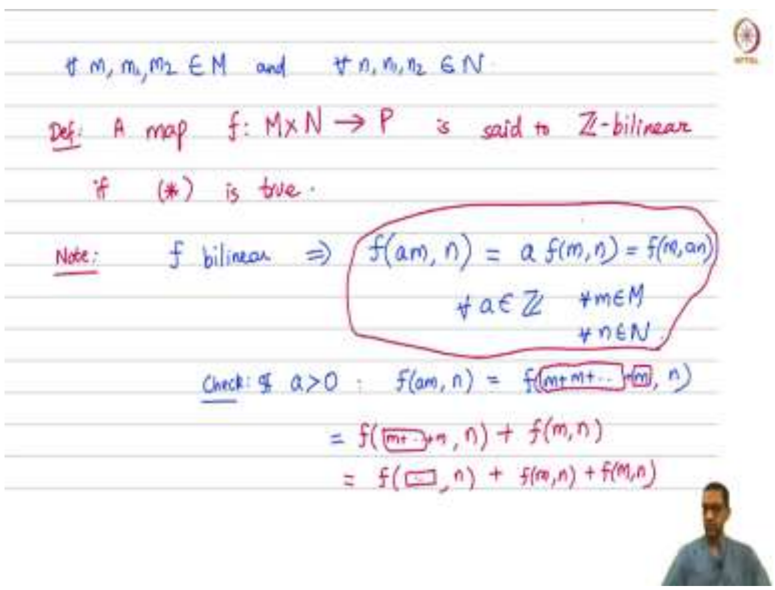

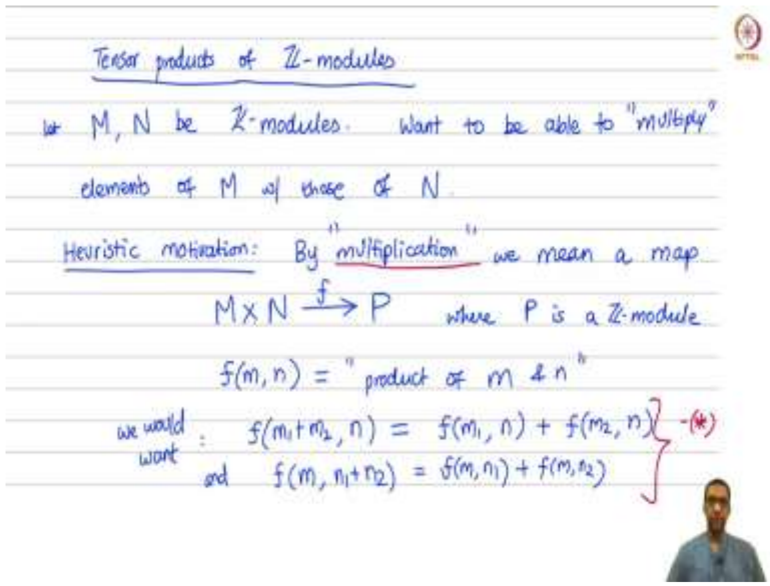

So, now you can see how this proceeds by induction. At every step you knock out one of the m's and so finally it gives you some of a such terms and this occurs a times, and that by definition is how you define. I mean that is how you define a times f m n this is not the Z-module structure in P. This is how you define the multiplication by a in the module P.

So, that is how we verify the first equality in one case so similarly you can verify the other cases as well as the second equality. So this is an important property so we will eventually generalize this to all rings in modules over commutative rings eventually but for the case of Z-modules you do not need to think of this as a separate axiom all you need is just this axiom star, just this bilinearity in, the linearity in each component will automatically imply this additional axiom.

This additional property in this case. Good, now, so what we have said so far to summarize is that when we say multiplication what we have in mind really is a bilinear map from M cross N to some module P. Now the point is there are many, many multiplications you can define because there are many choices of P, there are many choices of maps and so on.

So, you can imagine that there are many, many ways of defining multiplications. So, that is the next point to keep in mind there are many, many multiplications if you wish, let us say it like that. In fact here is the, I mean multiplication is a bilinear map, in fact observe that if I have one multiplication that is I have one bilinear map from M cross N to some module P then I can construct from this other multiplications as follows.

What can you do? Let us pick another Z-module Q, so P was the Z-module, Q was a Z-module and suppose I have a homomorphism we will call it phi so let Q also be a Z-module and phi, let phi me a homomorphism, be R or in this case case it is just Z so phi be, so Z-linear map between Z-modules is just a group homomorphism.

It is a group homomorphism from P to Q. Now observe, so if suppose this is bilinear and phi is just linear, so this is just ordinary, linear map. Now when you compose these two maps, when you compose what do you get? You get a map from M cross N to Q. So, this is composition phi composition f. Now here is the important fact that this new map is also bilinear and the proof is rather easy if you just compute phi composition f.

So, what we need to check m 1 plus m 2 comma n is phi evaluated on f of this, but f is bilinear so it splits into two. Now phi is linear which means that phi or a plus b is phi of a plus phi of b. So, this is just phi acting on f plus phi f. so this composition of m 2 n, and that is exactly what we had to prove. That is the bilinearity in the first component.

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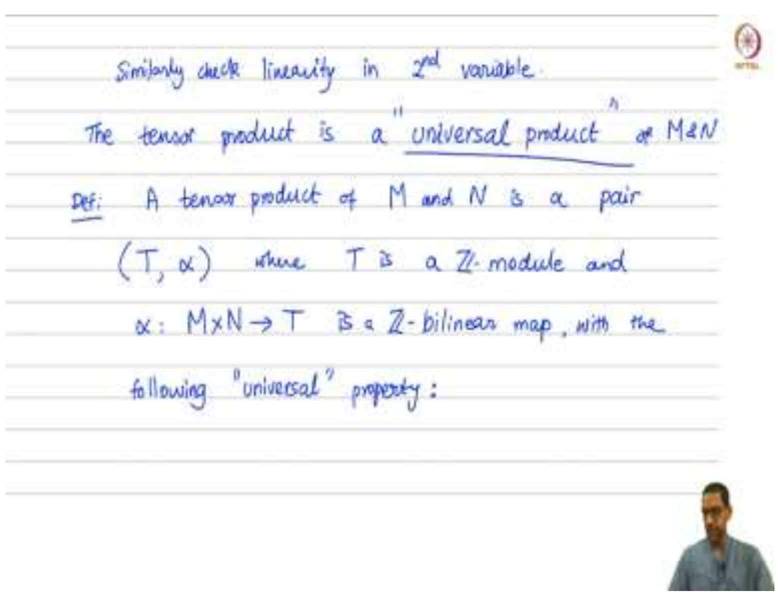

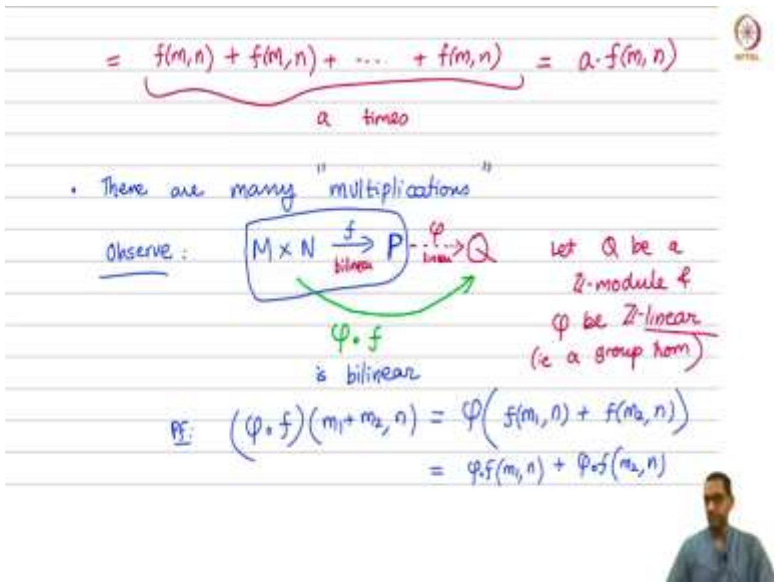

Similarly you have to check linearity in the second variable. So, sometime we say first variable and second variable when we mean the first input and the second input M and N. So, what does this mean? It means that there are, once you have one bilinear map you can construct other bilinear maps and in some sense what is the tensor product? It is sort of the most universal bilinear map.

So, the tensor product is a universal multiplication or it is a universal product if you wish of M and N. Now what does the universal product mean, the phrase? It is a product from which all other products can be formed via this method. So, somehow the tensor product is sort of a most atomic sort of product you can form, the most universal one.

Once you form the tensor product any other product of M and N you can form by taking the tensor product and sort of composing it with one additional linear map. So, that is what the tensor product is. So,mehow it is the most optimal product. So, let me just go ahead and make the formal definition here. Definition, a tensor product, because at the moment we do not know that it exists or is unique.

So, let me say a tensor product of M and N is the following. It is the pair, let us call it T comma alpha where what is T? T is a Z-module and alpha is a product meaning alpha is a bilinear map with the following universal property, and this property is usually called the universal property of the tensor product. Okay so what is the universal property?

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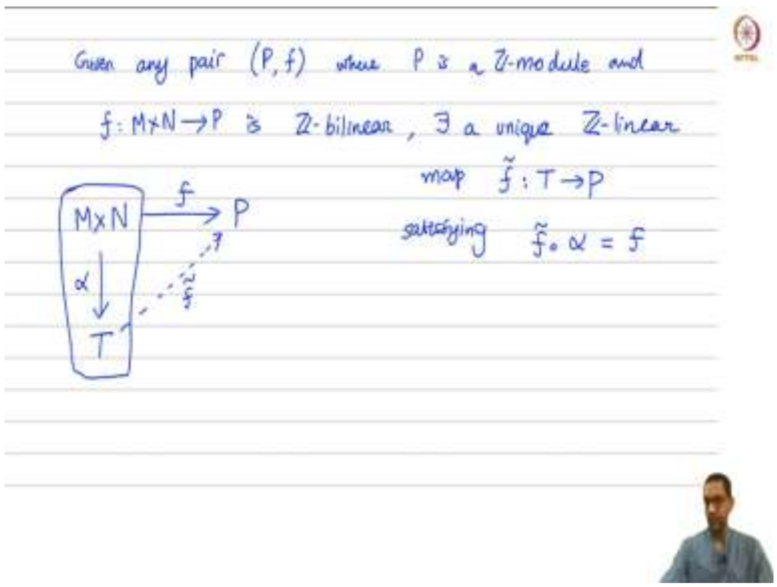

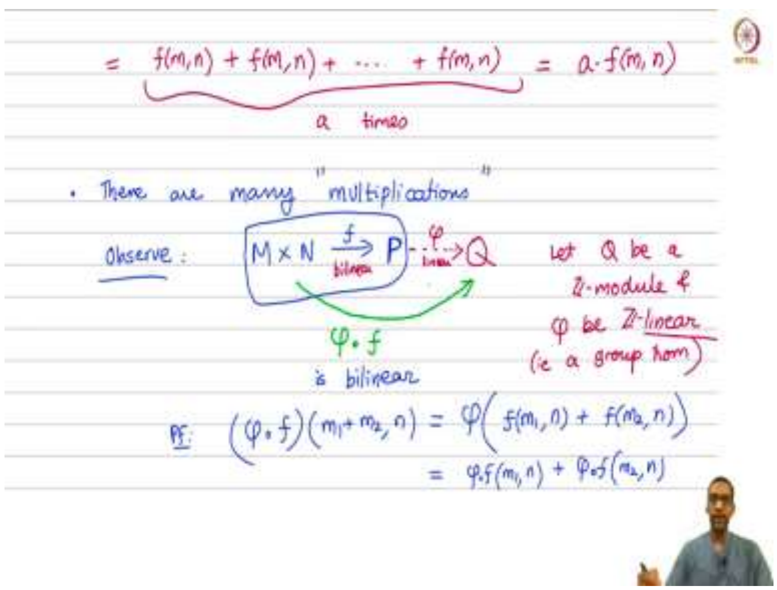

It says given any other such pair P f where what is P? As before it is a Z-module and f is bilinear. We usually depict all this by a diagram so let us draw the diagram so what are we given? M cross N to T there is a map alpha bilinear map, so this remember morally we are thinking of it like some sort of product.

Now, what we are saying is that tensor product is a very special guy meaning given any other product P and map f then what is true given any such pair then there exists a unique map, well what sort of map? There is a unique Z-linear map let us called it f tilde or f tilde as a map like this from T to P satisfying the following property that this diagram commutes by which we mean if you follow the arrows in either of the two ways you get the same answer.

f is a same as f tilde composition alpha. Now what does this mean? If you just go back and look at what you said earlier that if you have one product then you can get any other product from it by further composing by a linear map. Now what this is saying is the tensor product, once you have the tensor product you can get any other product P comma f from it as follows.

You first do the tensor product map, the map alpha and then you further compose it by this linear map f tilde. So, the tensor product is somehow the most optimal product. You can get everybody else from the tensor product. Okay now of course at this point it is not clear that it exists and, well we have to do some work to show that it exists but one thing that we can certainly do right away is to show that it is unique.

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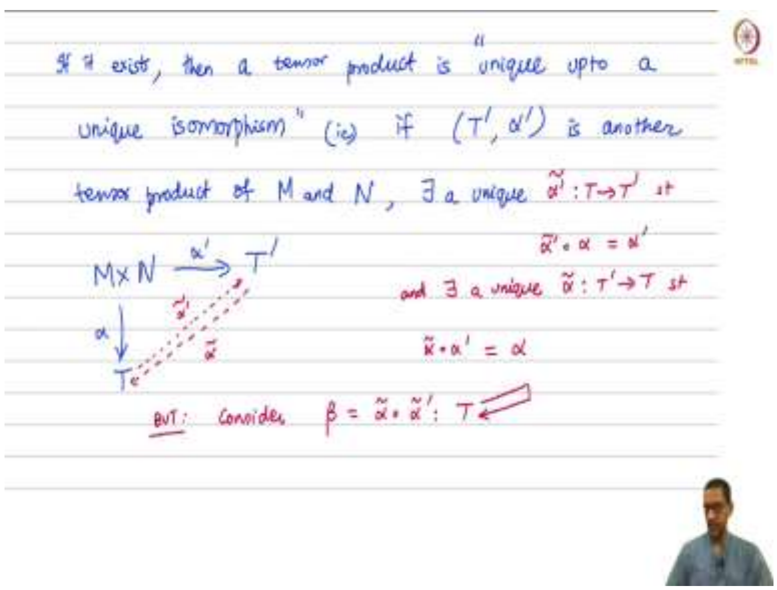

So, if it exists then, well the tensor product is unique in a very specific way. It is called unique up to a unique isomorphism. This is the usual phrase one uses. Now what does it mean? i.e. if I give you another tensor product if T prime alpha prime is another tensor product meaning it satisfies the same universal property. It is another tensor product of M and N. So, what does it mean?

So, I have M cross N to T. They also have another map like this then there exists a unique alpha tilde and beta tilde. Well we can use the defining property of the tensor product. So, let us just analyze for a minute what will happen, because I know that T comma alpha has the universal property so that is a tensor product.

Now, T prime and alpha prime think of it as just some product. So,me bilinear map. So, by the universal property of the tensor product there is going to be a map like this, which is what you would have called alpha prime tilde. So, there exists a unique map alpha prime tilde from T to T prime. Such that the diagram commutes, right?

Which means alpha prime tilde composition alpha is alpha prime, but by the same token I can invert the roles of T and T prime. In other words I can think of T prime alpha prime as it is a tensor product and T comma alpha is just another product. It is a bilinear map then by the same token I have a map alpha tilde, because T prime alpha prime is a tensor product.

So, I can also conclude there exists a unique, this is the Z-linear map, so similarly there is a unique map alpha tilde, Z-linear from T prime to T, such that alpha tilde composition alpha prime equals alpha, meaning the diagram commutes in either of the two directions. But now here is the interesting thing.

Now observe what happens if you sort of compose these two maps, but, so let us to the following, let us look at this composition. Consider the composition of the two maps. So, let us call it something beta equals alpha tilde composition alpha tilde prime. Now, what is this? This is a map from T back to T like it goes first to T dash and comes back to T. Now we have the following diagram. So, we can actually draw a different diagram. I have M cross N.

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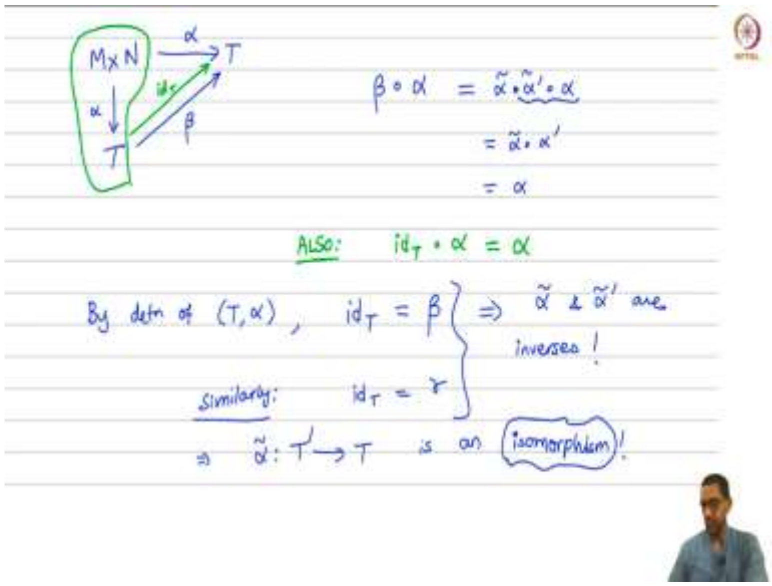

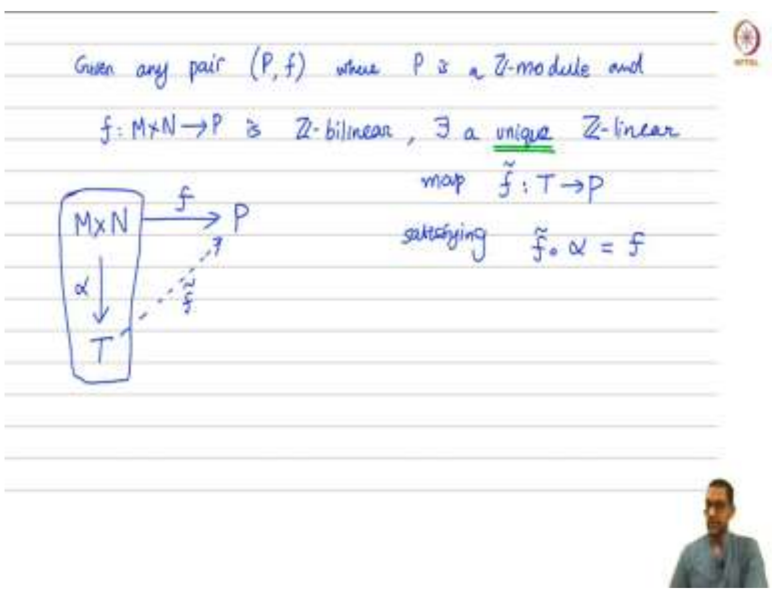

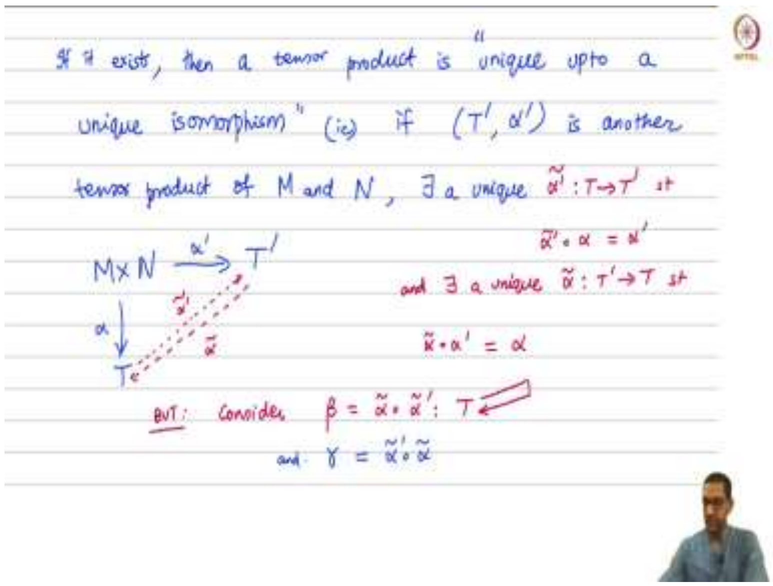

Yeah, I have alpha now again I put T in both places. Now observe that on the one hand the, so now I have this map beta here. So, I claim that this diagram also commutes, meaning beta composition alpha is equal to alpha. So, let us calculate. So, what is beta composition alpha?

Now, let us go back beta is alpha tilde composition alpha prime tilde, and then I have to compose it with alpha. Now let us go up again what is alpha prime tilde composition alpha, well that is alpha prime. So, this is alpha prime tilde composition alpha. It is just alpha prime by its defining property. Now let us go up again. What else do we know?

Alpha tilde composition alpha prime is alpha. Alpha tilde alpha prime is alpha. So, what we conclude is that because sort of the diagram's commuted in both directions in fact this diagram commutes. So, what it means is the beta composition alpha is the same as alpha, but observe that there is actually another obvious map which makes this diagram commute.

What is this map? Well that is just the identity map on T. also observe that the identity composition alpha is also alpha. So, I have two different maps, both of which make this diagram commute. I mean I have two maps but remember what the tensor product was supposed to satisfy. So, what is the definition of a tensor product?

Given any such pair P comma f there exists a unique Z-linear map which makes this diagram commute. Now beta is of course a Z-linear map because it is a composition of two Z-linear maps. Now the point is that I now will apply the definition of the tensor product to this diagram.

In other words I think of this as my tensor product and now think of T comma alpha as just being some product, some bilinear map is given.

Then by the universal property of the tensor product I know there must be a unique map from T to T which makes this diagram commute, but I know there are two maps. So, those two maps must be equal. So, by the definition if you wish, by the definition of T alpha, in other words by the universal property the uniqueness tells me that these maps must be equal to each other.

What does that mean? Well it just says that alpha tilde and alpha prime tilde are inverses of each other. I mean I should also do a similar thing in the other direction which is with, so I should probably have said consider also the other guy gamma which is the thing in the other direction. So, which is alpha, first go along alpha tilde and then go along alpha prime tilde, and this, but whatever I am telling you about beta also applies to gamma.

So, I should also do that, it is going to be similarly applying the same logic and conclude that the map gamma is also identity. Now, this tells you that if you compose alpha tilde and alpha prime tilde in other order you get back the identity so they are actually inverses of each other. So, what that means in particular is that these two are isomorphic. They are, they give you isomorphisms between T and T prime. So, what is alpha tilde?

It was a map from T to T prime, is an isomorphism. So, that is what it means by saying to unique up to unique isomorphism. We know that there is always a map, like if I have another tensor product I know that there is certainly a map from one to the other. Now what I know is that, sorry! Alpha tilde is a map in other direction T prime to T. So, I know that there is a map in one direction, I mean either direction but what I also know now if both of them are tensor products then that map must actually be an isomorphism.

So, this is the key thing about tensor products, the definition itself. If it exists then it is, there is an isomorphism between them and that isomorphism is unique.

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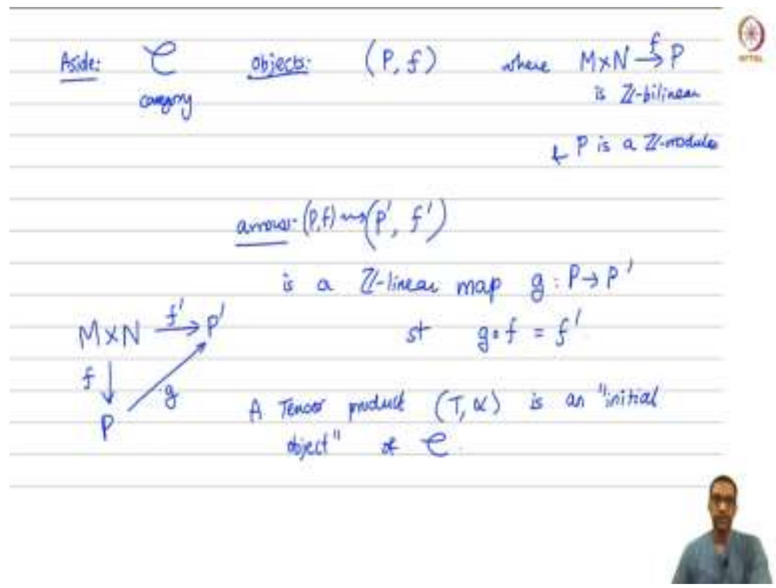

Now, let me just make the quick aside on categories and so on since you have sort of learned a little bit about categories and functors. Another way of putting all this is to say let us take the category C whose objects, the objects are now slightly more complicated. So, what are the objects? The objects are pairs P comma f where this is Z-bilinear. You take all Ps and, and P is a Z-module.

So, take all Z-modules P with bilinear maps from M cross N to P that is your objects in your category, and what are the morphisms or arrows in this category? Well the arrows are maps between the P's okay so given a P and a P prime so if they give you another object P prime f prime what is an arrow from P f to P prime f prime so you should say an arrow from this pair to this is Z-linear map.

Well what should it be? There is only one obvious thing one can do so given M cross N to P and given M cross N to P prime what can I do? Well I can define an arrow to be a map from P to P prime which makes this diagram commute. So, let us call this something. What is an arrow? It is a map, maybe g, so Z-linear map g from P to P prime such that g composition f is f prime.

You sort of define it in a way that is reminiscent of how that tensor product was given. So, you can define a category in this way, pairs P f, with arrows being maps between the P and the P dash which makes this diagram commute. Once you set things up in this way, then observe that the tensor product is just an initial object in this category.

So, then the tensor product or a tensor product T alpha then becomes what we had called is an initial object of the category C, and recall an initial object just meant it is something from which there is a unique arrow to every object in the category and that is, if you see that is just another way of restating the way we define tensor products.