Algebra-II

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Lecture 58

Categories: Fourth Problem Session

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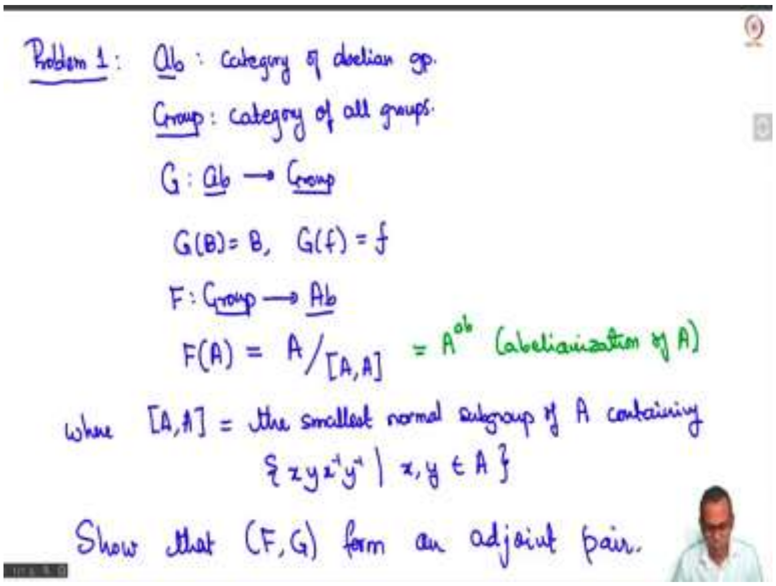

Let us solve some problems involving adjunctions. So, here is problem 1, let ab denote the category of abelian groups. So, the objects are abelian groups and arrows are just group homomorphisms and let Group denote the category of all groups, and let us define G from ab to Group to just be the usual inclusion functor that is given an abelian group B you take it to B itself which you think of now as a group. Forget the fact that it was abelian.

And given a group homomorphism, so G B is equal to B and G f given a group homomorphism f you take it to the group homomorphism f, and you define f from Group to ab as the abelianization functor. So, f of a group A is A modulo its commutative subgroup. So, let me remind you what this is. It is the smallest normal subgroup of A containing the set of all commutators i.e. the set x y x inverse y inverse where x and y are in A.

This thing is usually called the abelianization of A. So, here is your exercise. Show that F comma G form an adjoint pair. So, it would be a good idea to pause your video at this point and go back and look at the definitions from the lectures and try to set up exactly what you need to proof in order to establish this and then do it. If you cannot do it then you can watch me solve it.

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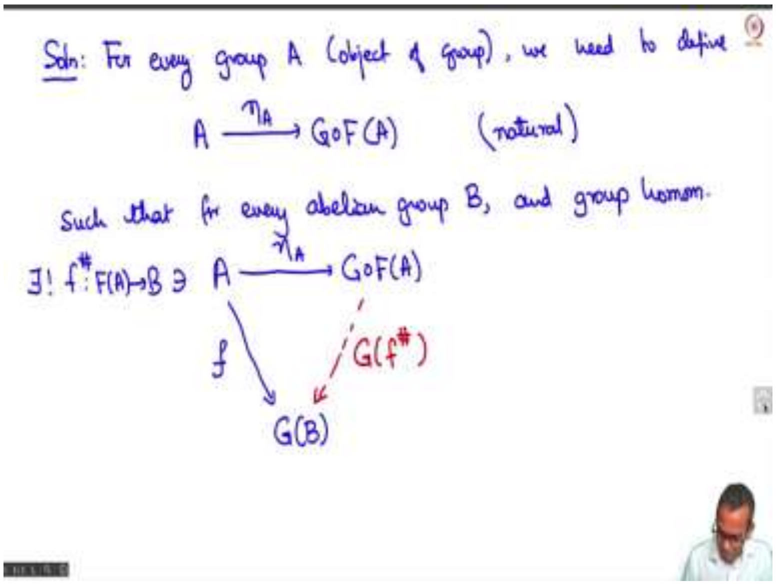

So, here is the solution. So, what we need is to setup the following thing, so for every A group that means object of group we need to define eta A to G circle F of A such that for every object B of abelian group and every group homomorphism A to, well we want to say B here but we are saying G B which is that we are just thinking of B.

Now, it is abelian group but we are just thinking of it as an object in the category of all groups. So, for ever group homomorphism here we have, there exists unique f sharp from F A to B such that this diagram commutes. So, this is G of f sharp. And this eta needs to be natural. That means this collection needs to give rise to a natural transformation. So, let us just decode what all these symbols mean.

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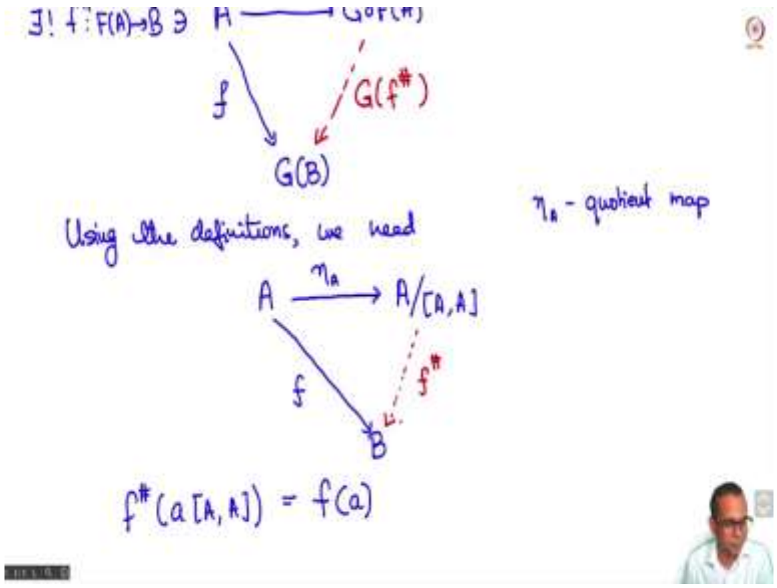

So, what we have is, so this diagram A is just now it is a group and F of A is, well it is the abelianization of the group so A mod A A and we need eta like this. Now this is, technically this is, well this is going to be an arrow in the category of groups right? So, this is an abelian group you can check but when we just think of it as an arbitrary group so we need an arrow like this and what else but just take the quotient map.

You need to check that it is natural but that is not but that is not difficult and I am going to leave it out, and what are you saying is that for every abelian group B whenever you have a group homomorphism from A to B that means every group homomorphism from A to an abelian group factors through the abelianization of A.

So, what you are saying here is there is a group homomorphism f hat from A mod A A to A and this is actually very easy to see. You just define whether it is forced f hat of a the coset A A to be f of a and you need to check that this is well defined and that it is indeed a group homomorphism from A mod A A to B, but there are all standard and I will leave it to you to work out all the details.

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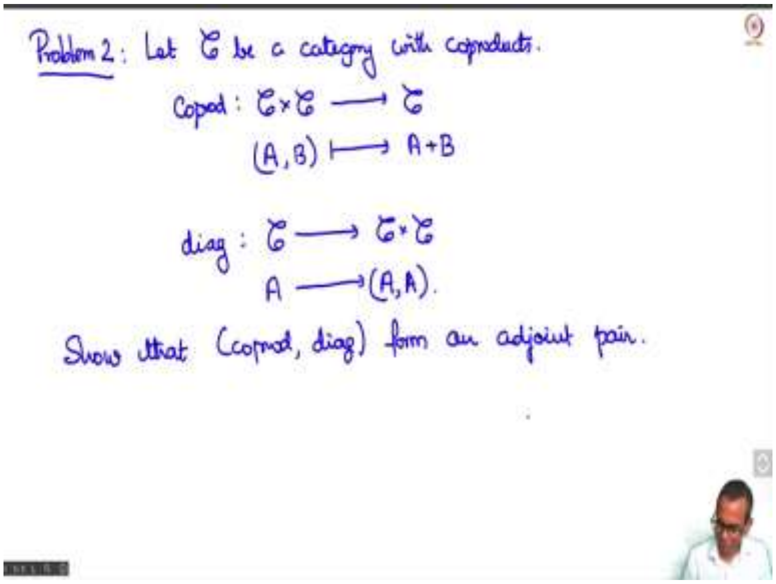

Let us look at problem 2. Let C be any category where co-products exist which means that given any two objects A and B you have a co-product which we denote by A plus B, right? Now we have a functor Coprod we can think of co-product itself as a functor. It goes from C cross C, the product to the category C with itself. So, its objects are pairs of objects in C to C.

And what does this functor do it takes A comma B to A plus B and you can work out what it does on arrows as well. If I have got pair of arrows in C it will define arrows on the co-products and then we have a functor called diag from C to C cross C and this is the functor which takes any object A to the co-product, no, not the co-product to the pair A comma A.

And the problem is show that Coprod comma diag form an adjoint pair. So, you should again go back to the definitions and try to see exactly what you need to prove and try to do it yourself, but if not you can watch me solve this.

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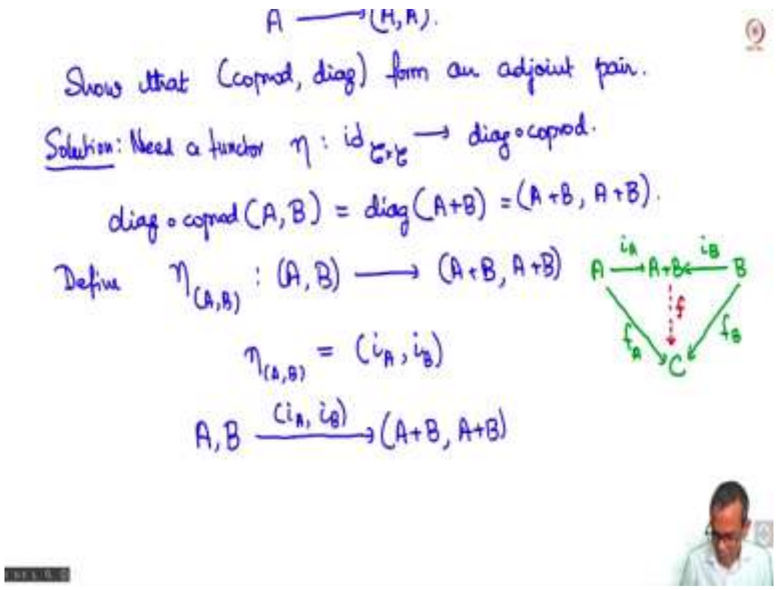

So, again we need to setup this eta from the identity functor of C cross C to the functor diag circle Coprod, this is what we need. So, let us see what this diag circle Coprod is. So, diag circle Coprod of, so this is a functor from C cross C to C cross C so we will give it an input of two objects from C, now Coprod takes this to A plus B and then diag takes it to A plus B comma A plus B.

So, what we need is eta A comma B from an arrow in C cross C from A comma B to A plus B comma A plus B, and how will we do that? Well you just define it by, so remember when you have a co-product, let me just recall for you the universal property of the co-product so when you have a co-product of A and B A plus B then you have an arrow from A to this co-product which we call usually i A you have an arrow from B to this co-product which we call i B.

And it has the universal property that whenever you have arrows from A to some object let us say C let us call is f A and you have another arrow from B to the object C then you have a unique arrow which we will call f from A plus B to C. So, in particular this co-product comes with arrows, i subscript A and i subscript B from A and B.

So, we can eta A B to be the most natural thing namely i subscript A comma i subscript B. i subscript A is from A to A plus B, i subscript B is from B to A plus B, and what we have now is

A comma B, we have eta A B so this is i subscript A i subscript B and we have A plus B comma A plus B.

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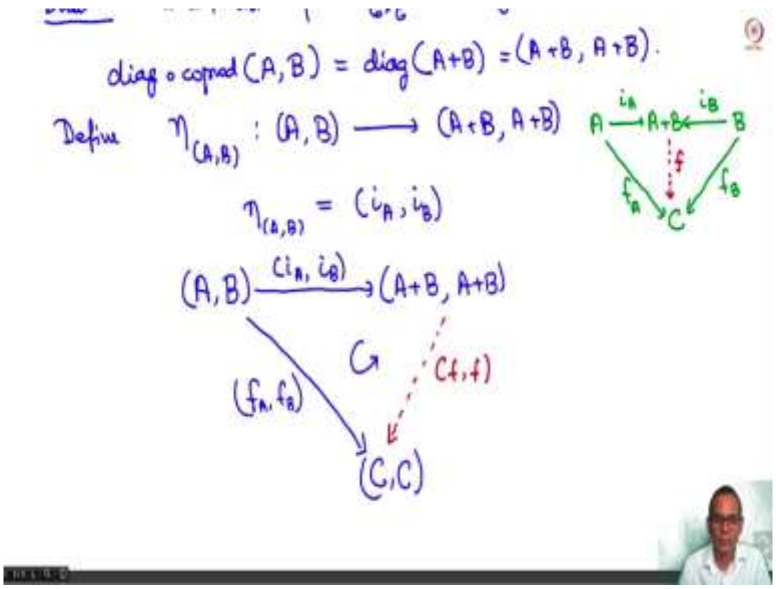

So, this is diag circle co-product of A comma B and we want to show that whenever we have an arrow from here to diagonal of C so diagonal of C is going to be C comma C for any object C of the category C. So, whenever we have this, well ever arrow here from A comma B to C comma C is going to be of the form f A comma f B, because it is just, the arrows in the category C cross C are pairs of arrows in the category C so an arrow from A comma B to C comma C is going to be a pair.

One arrow f subscript A from A to C and another arrow f subscript B from B to C, and when you have this it is easy to figure out what to put here. You just put f sharp to be f, you just take f. Now if you take f and you take diag of f so diag of is just f comma f where f is as defined by this universal property of A plus B using the arrows f A and f B. Then from this universal property it follows that this whole diagram commutes.

Now, I leave it to you to work out all the details but this proves that the diagonal and co-product functors are adjoints.