Algebra-II
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Lecture 57
Adjunction

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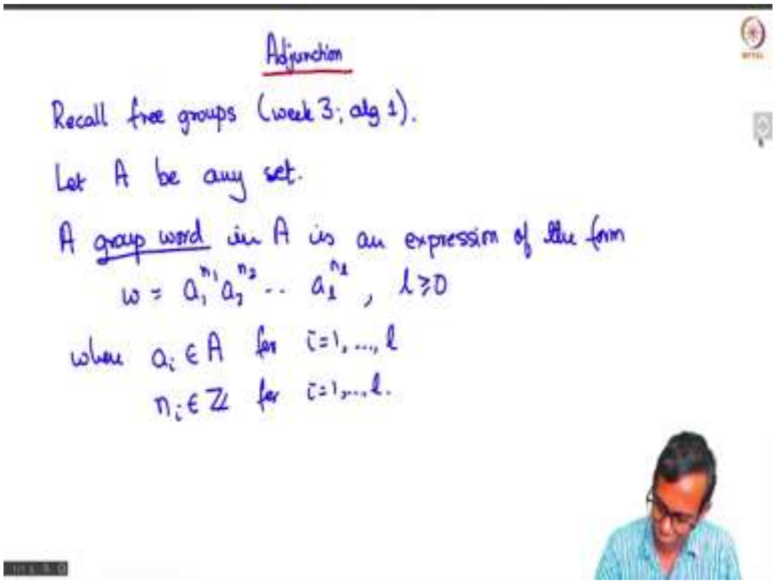

In this lecture we will study Adjunction which is perhaps the most important construction that comes from category theory. I will start with an example which seems to have nothing to with category theory which is the definition of a free group. So, recall free groups. You can look at week 3 of Algebra-I and how it works is this. So, let A be any set and a group word in A.

I am using the phrase group word because if I just use words, I will get what is called the Free Monoid but I want to construct the free group. So, group word in A is an expression of the form W equals a 1 to the power n 1 a 2 to the power n 2 a l to the power n l, where l is any non-negative integer. I allow l to be 0 in which case I will be looking at the empty word where each a i comes from the set A and n i is an integer.

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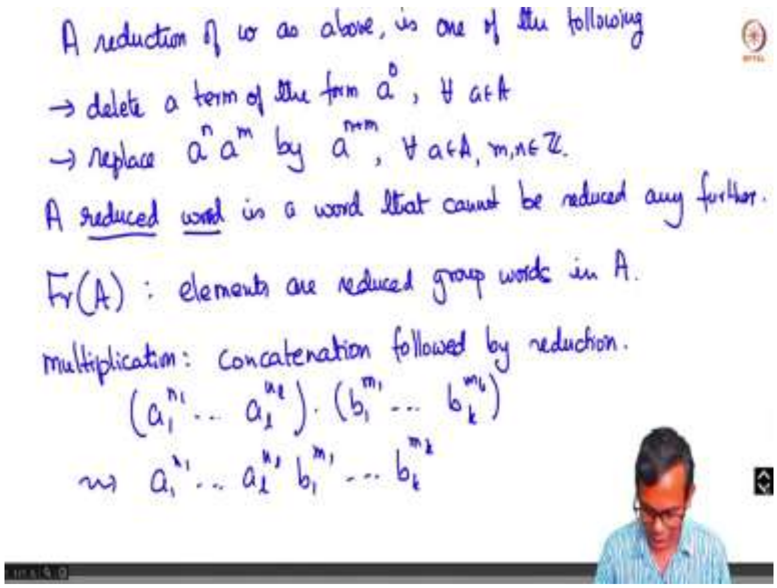

And a reduction of a word say as above, is one of the following steps. So, what you do is you can delete a term of the form a i to the, a to the power 0. So you can, if any of the exponents in this W is 0 you can just delete that term and the second one is you can replace a to the n a to the m by a to the n plus m for any a in A. So, this is for all a in A and this is for all a in A, m n in Z.

So, basically what we are saying is if two consecutive terms in your word have the same latter then you can club them and replace the exponent by the sum of the two exponents. So, this could be positive or negative but you add them up as integers. So this called Reduction and note that each time you reduce a word you will be reducing l itself.

You are reducing the number of the terms in the words and so this Reduction process will eventually stop and, so every word can be reduced till it can no longer be reduced any further. Maybe you will get the empty word but certainly you will reach a stage when you cannot reduce it any further. So, a reduced word is a word that cannot be reduced any further.

And now the free group of A is defined as follow. Its elements are reduced group words in A and its multiplication operation is a concatenation of words followed by reduction. So, if you have two words a 1 to the power n 1, a 2 to the power n 2, a l to the power n l and say you have another word b 1 to the power m 1, b k to the power m k then that concatenation is this word a 1 power n 1, a l power n l, b 1 power m 1, b k power m k.

Now this word, this concatenated word may not be reduced. It could be for example that a l and b 1 are equal in which case you have to club these two terms to get a reduction and after you reduct, club them, it could happen that m 1 is minus n l in which case those things cancel out and so on. So, you may have to do a series of reductions but at the end of the day, you will get a reduced word and that will be an element of the free group on A and so that will be the product of these two words.

And you can check, it is not very difficult to check that this actually gives rise to a group. These things are done, maybe in a slightly less generality in Algebra-I. This is also related when we studied a co-product in the category of groups.

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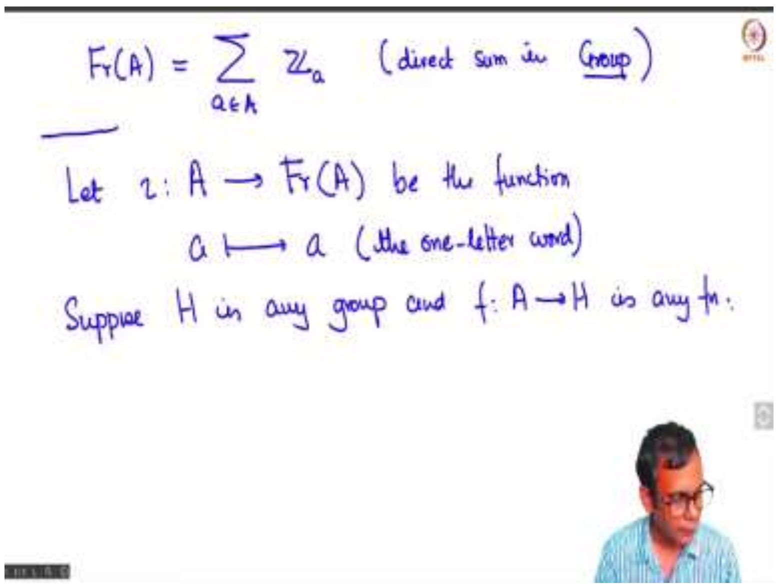

So, it turns out that free group on A is actually a in A. So, this is just integers, one copy for each A. This is a direct sum in the category of groups. How does this work? So, we saw finite direct sums, it was what we called an amalgamated product. So, let iota from A to free groups on A be the function which takes a to a where this is to be the thought of as the one letter word, consisting of just one letter namely A. Now, suppose H is any group and f from A to H is any function.

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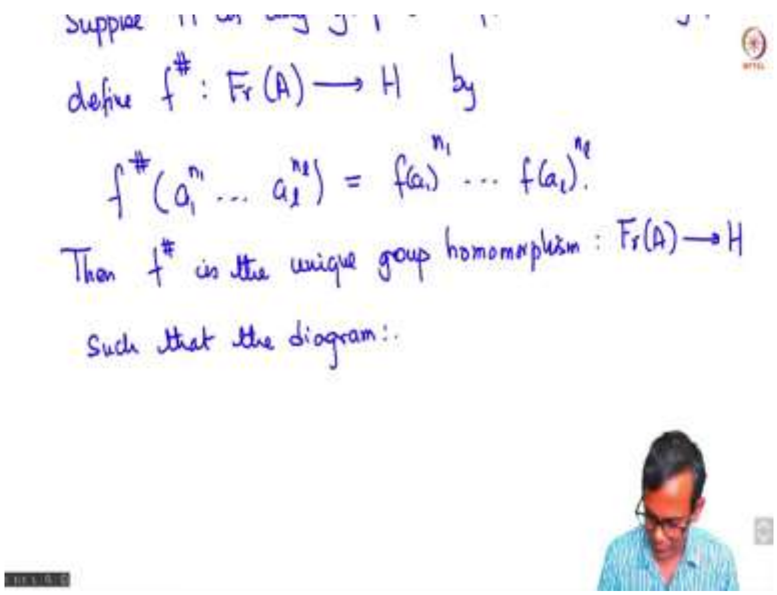

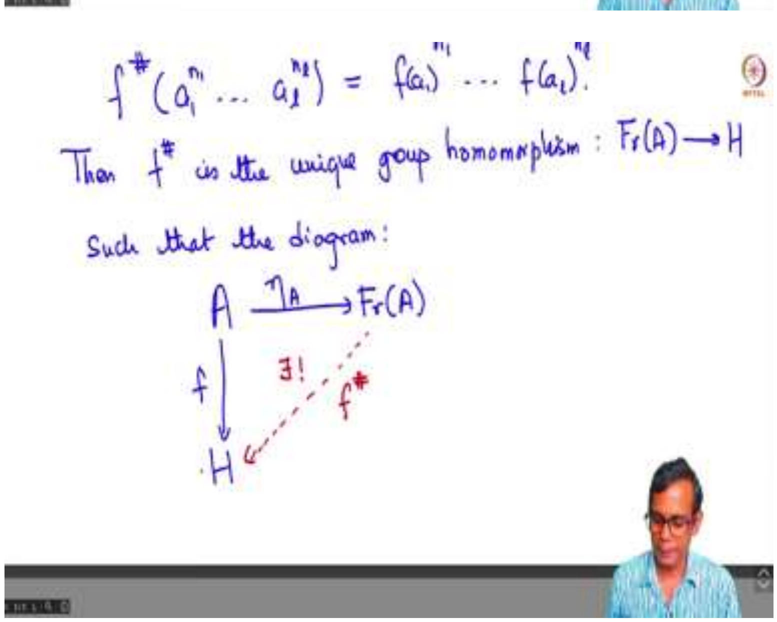

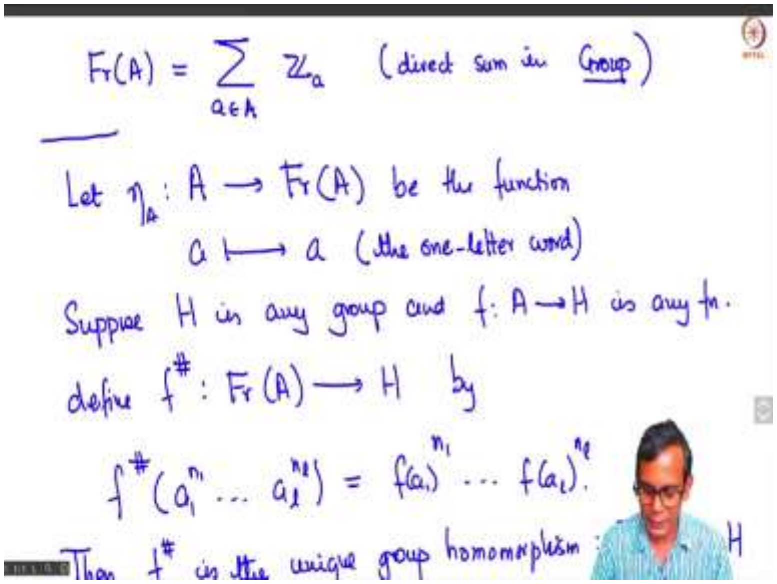

Then we can define f hash from the free group generated by A to H as follows. So, I need to define f hash on every reduced word. So, a 1 to the n 1 a l to the n l will be, so now f is just a function so I have f of a 1 that is an element of H and I will raise it to the power n 1, f a l to the n l and it turns out that f hash is the unique group of the homomorphism from the free group on A to H such that the diagram, So, let me draw this diagram.

We have A, we have, so let me just do a slight suggestive change of notation here. Let me call this instead of iota, let me call this eta subscript A. You will see that it is a natural transformation. So, we have A, eta subscript A to free group of A and then we have any function f from A to H any group and what we are saying is that there exists a unique group homomorphism f hash from free groups of A to H. So this is a universal property of the free group on A,

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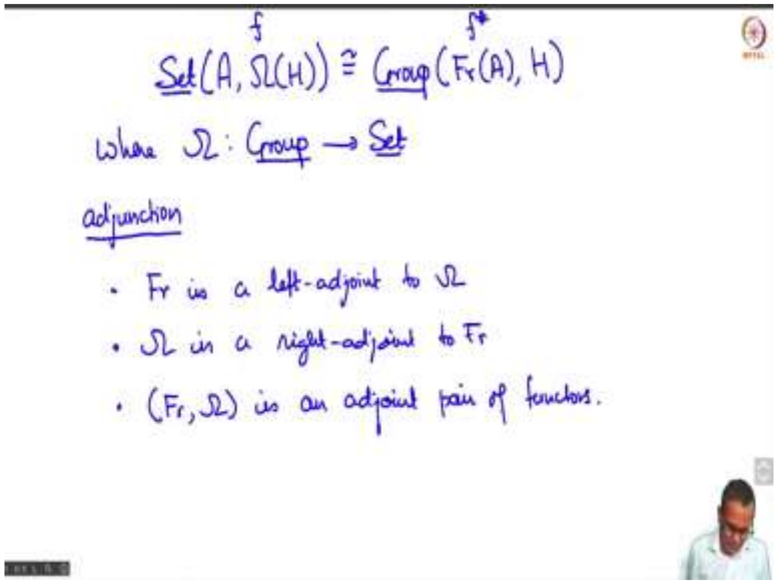

Viewed in another way we have that set homomorphisms or arrows in the category of sets from A to omega H is the same as group arrows from free A to H where omega denotes the forgetful functor from the category of groups to the category of the sets. Now this kind of situation is known as Adjunction.

What we say is that Fr is a left-adjoined to omega or we may say that omega is a right-adjoined to Fr or we may say that Fr comma omega is an adjoined pair of functors. So, here note that Fr occurs on the left and omega occurs on the right and you are allowed to go for A, morphisms from A to omega H. This is what we called f and arrows from free A to H, this is what we called f sharp and this was a adjective correspondence.

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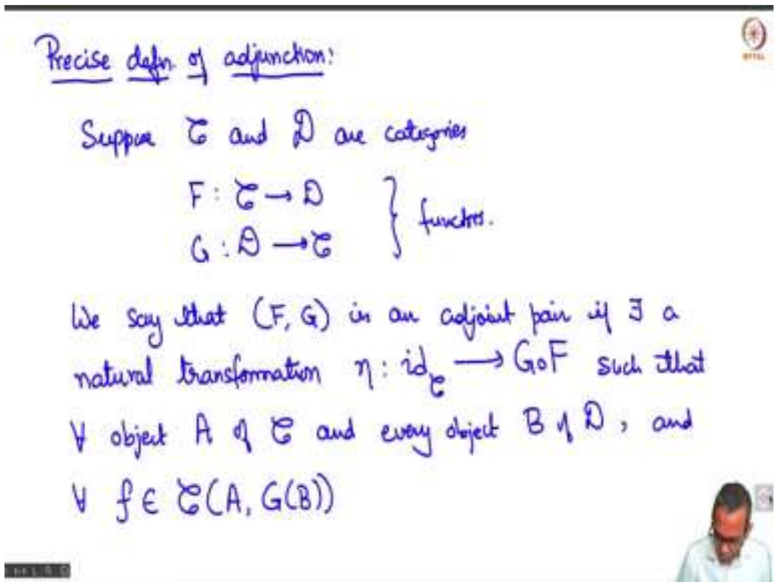

So, let me make this idea of adjunction more precise. So, the situation is as follows. Suppose C and D are two categories and you have functors F from C to D and G from D to C, functors. Then we shall say that F comma G is an ad-joined pair or as I explained before we will say that F is a left-adjoined of G or we could say that G is a right-adjoined of F if there exists a natural transformation, eta from the identity functor from C to C to the functor G circle F.

So, both these are functors from C to C such that for every object A of C and every object B of D and for every arrow f in the category C from A to G of B we have there exists a unique arrow f sharp in the category D.

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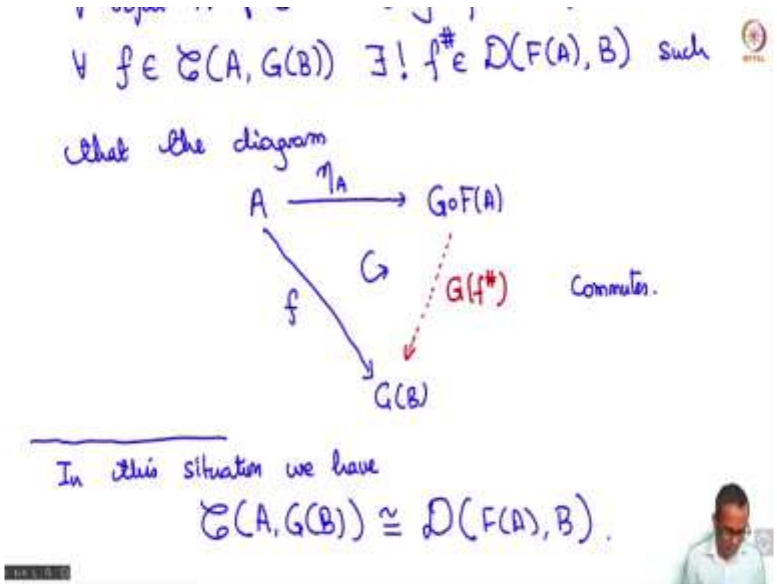

And now we would have F A to B such that the diagram. So, we have A and then we have G circle F of A and from this object to this object and see we have the arrow eta A coming from the natural transformation eta and given f from A to G of B we are saying that there exists a unique f sharp such that if you put G of f sharp here, not f sharp itself but if you put G of f sharp here then this diagram commutes.

This sets up a bijection. I should say there exists a unique f sharp. So, the uniqueness is also required. So, this sets up the bijection between f and f sharp because f sharp exists uniquely if f is given and so that is the function in one direction and the other direction given f sharp you can take G of f sharp and compose it with eta A and so you get f and so what this is saying is, so this is a complete set definition but what this is saying is we have, in the category C the arrows from A to G B are in bijection with the category D. The arrows from F A to B.

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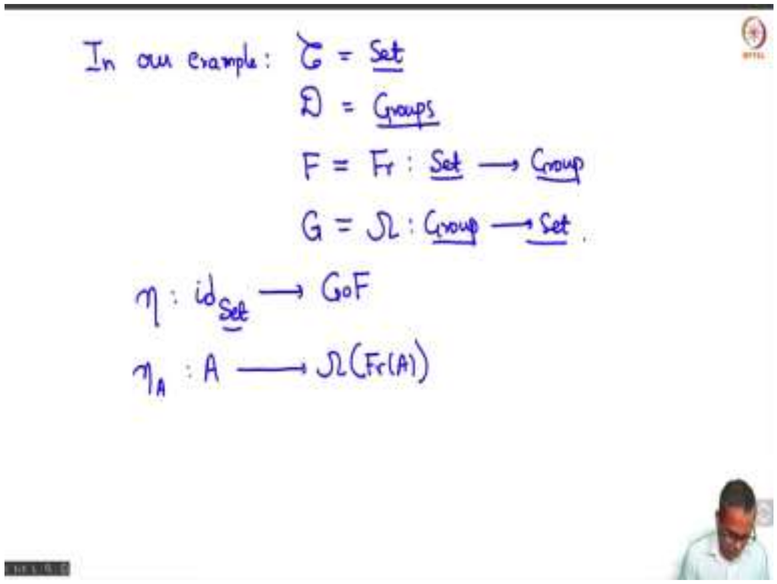

In our example we had taken C to be the category in the sets, D to be the category of groups. We have take F to be the free group functor. This is from sets; given a Set it associates a free group to it. G was the forgetful functor from group to Set and eta is the natural transformation from the identity functor from the category Set to itself to the functor G circle F and what is this?

So, given the set A, this eta A goes from this to omega of free of A. In other words eta A is just a function of sets like a function with no further requirements or being a homomorphism or anything from A to the set underlying the free group generated by A. So, it is just a function from A to Fr A and the universal property for the free group is exactly the statement that Fr and omega form an ad-joined pair.

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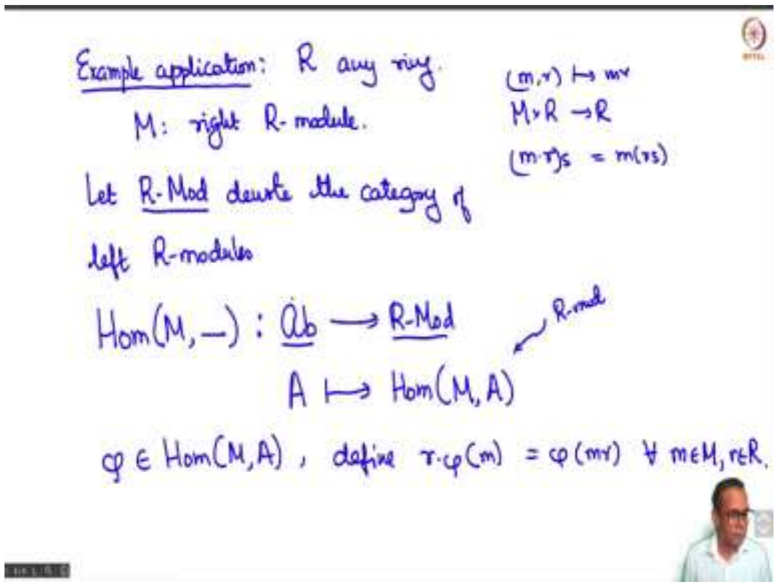

Now the notion of adjunction is very useful. We will see later in this course that suppose you are given a ring R. So, example application which you will see in detail later in this course, so let R be any ring and let M be a write R-module. So, that means there is a function from M cross R to R, satisfying various conditions in particular m r so this is usually known m comma r goes to m r and m r s is m r s, among other conditions.

So, we are saying that m is a right R-module and let us consider the category of left R-modules. Now there is a functor I will call it Hom M comma dash. This is a functor from the category of abelian group. So, this is the category of abelian groups to the category of the R-modules left R-modules which does the following. It takes an abelian group A and maps it to group homomorphisms from M to A.

So, if you have phi belongs to Hom m a, you can think of phi define r phi to be, so for every r in the ring R you can define r phi of m to be phi of m r for all m in M, r in R. And this makes Hom M A an R-module. So, in this way a Hom M dash defines a functor from the category of abelian groups to the category of left R-modules. You still need to think what it does at the level of arrows but here is what it does at the level of objects and this functor has a left ad-joined

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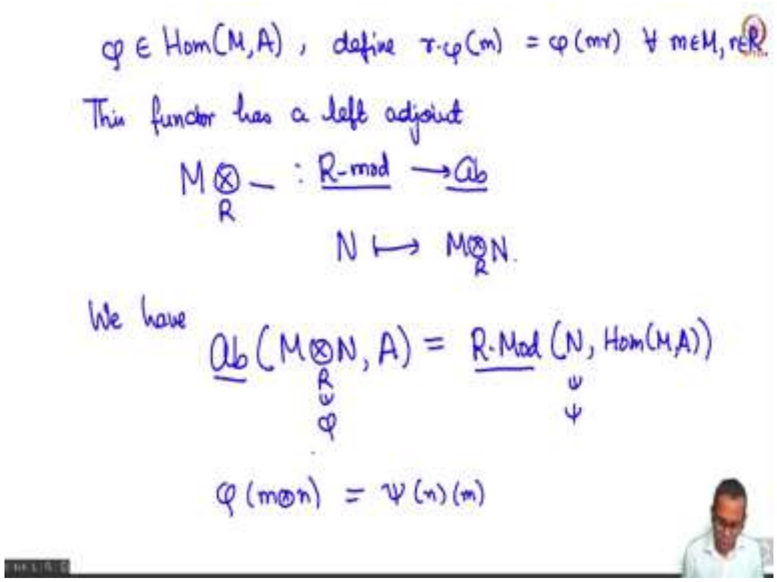

And that is a functor called tensor with M over R. This is a functor from the category of R-modules to the category of abelian groups and what it does is it takes in R-module N and maps it to the abelian group M tensor over R N. This will be explained later in the course and so the adjunction here says that the arrows in the category of abelian groups from M tensor N to any abelian group A is the arrows in the category of R-modules from N to Hom M A.

And you can, well if you know a little bit about tensors what it does here if you have a phi here and psi here then they are related by the following equations. So, psi of n of m, maybe I will write psi on the right, phi of m tensor n is equal to psi of n of m. You can use this identity to define phi in terms of psi and psi in terms of phi.