**Algebra-II**
**Professor Amritanshu Prasad**
**Department of Mathematics**
**The Institute of Mathematical Sciences**
**Lecture 56**
**Categories: Third Problem Session**

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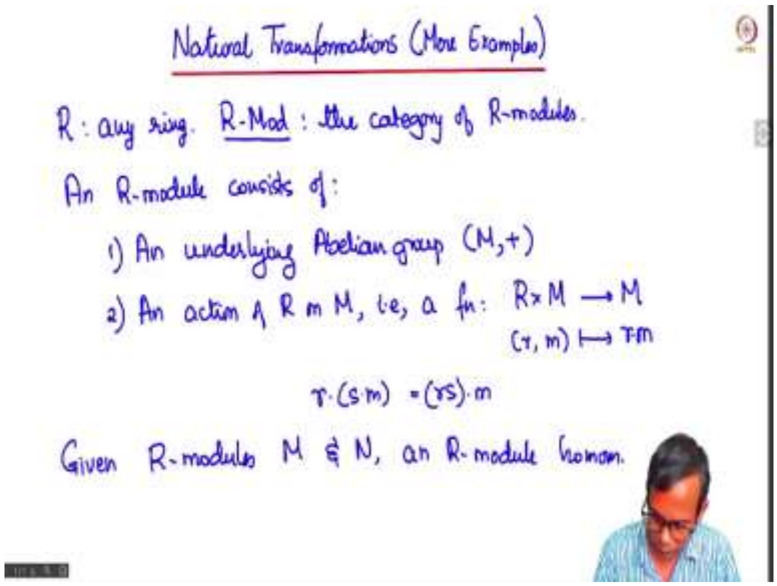


Let R be any ring and we will use R-Mod to denote the category of R-modules. Recall that in R-module consists of firstly and underlying abelian group M and its group operation is usually denoted by addition and secondly an action of R on M, i. e. well this is not a group action. It is a ring action. I. e. function R cross M to M where this is an Abelian group homomorphism.

You can think of R cross M as an abelian group and this is a function which we usually denote by r comma m goes to rm and it must have the property that r dot s dot m so if I use dot here is rs, this is multiplication in r dot m, so sort of associativity property.

So, these two pieces of data give you an R-module and the category of R-modules consists of, its objects are all R-modules and its arrows, so given two R-modules are M and N then R-module homomorphism from M to N is a function such that.

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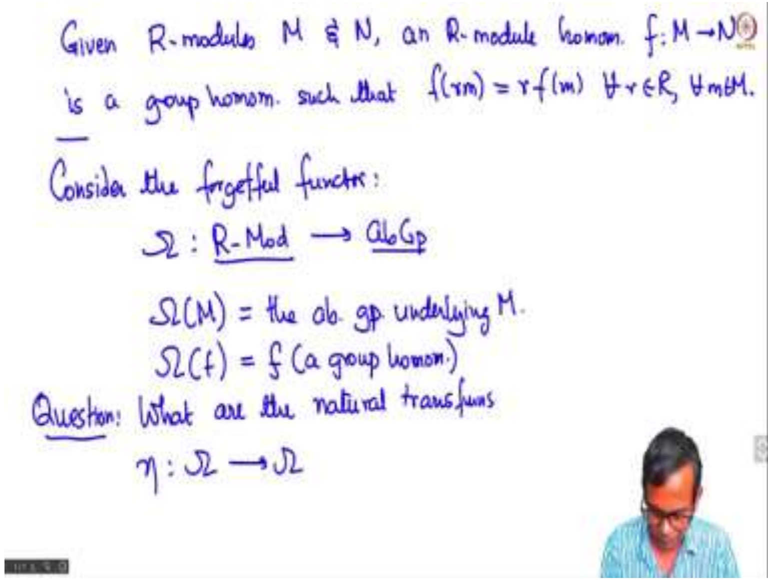


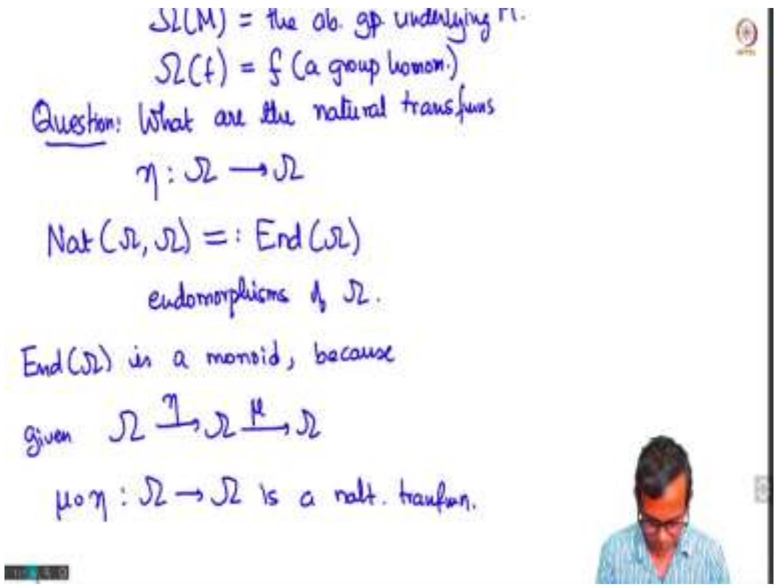

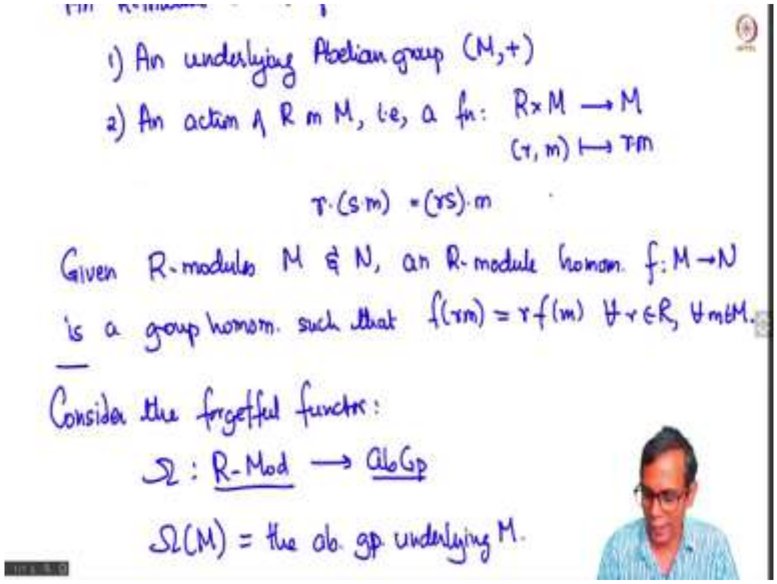


Well, it is actually a group homomorphism of abelian groups M to N such that f of rm is rf m for every r in R and for every m in M. So, basically it respects the actions of R on M and N, and so the category of R-modules has its arrows R-module homomorphisms.

Now, there is a functor omega, the forgetful functor which I will denote omega from the category of R-module to the category of abelian groups. So, this category of abelian groups is the category whose objects are abelian groups and arrows are group homomorphisms, and what this functor does is that given an R-module M omega M is just an underlying abelian group.

So, it is called forgetful because you are going to forget this second part of the data defining an R-module namely the action of R on M, and given and R-module homomorphism well we defined an R-module homomorphism to be firstly a group homomorphism and then it must have this additional property, that f of rm is equal to rf m so an R-module homomorphism is an abelian group homomorphism. So, we will just write f itself viewed as a group homomorphism.

So, that is the forgetful functor and the question that we shall address now is what are the natural transformations? Eta from omega to omega. So this set, this collection of natural transformations we can denote by Nat omega omega so we are asking what is Nat omega omega and sometimes this Nat omega omega is also denoted by End omega which means the endomorphisms of the functor omega. That is just terminology.

Now, note that End omega is actually a monoid because firstly we have the identity natural transformation from omega to omega and secondly if you have two natural transformations eta from omega to omega and mu from omega to omega then we can get mu circle eta is also a natural transformation from omega to omega.

And you can show that this composition law will satisfy the axioms for being a monoid. So, the question is can you describe this monoid endomorphisms of omega? You can try to solve this yourself. Pause the video now and try to see what to do. It is a bit tricky. So, you may need to look at the solution that I am going to provide.

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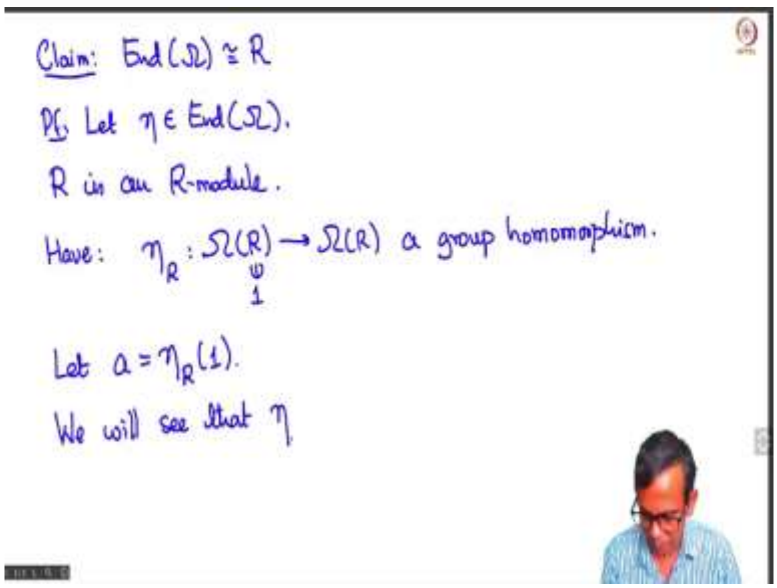

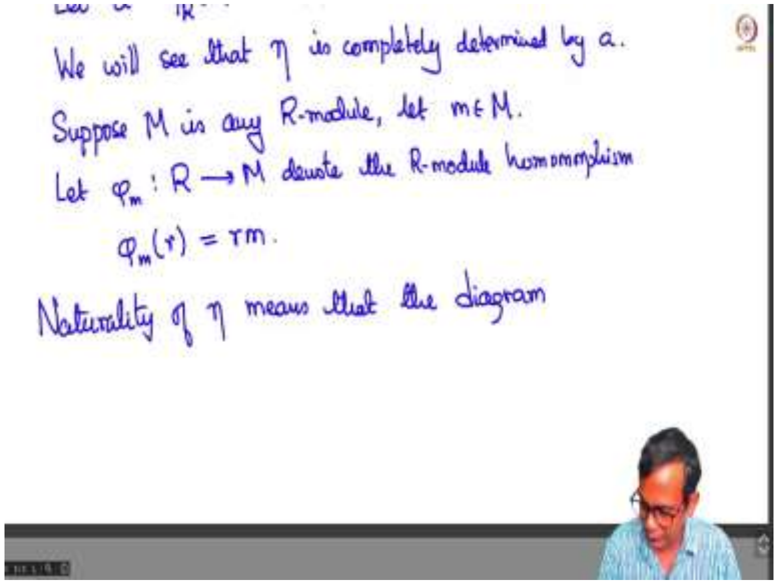


So, we want to compute End omega. Now I claim that End omega is the same as R. In fact End omega is not just a monoid but it is actually a ring. The monoid structure will be the multiplication structure on this ring and there will be an additive structure, but firstly let us just see how this works at the level of sets.

So, suppose you have a natural transformation from omega to omega and now the first hint as to how to solve this problem, now R itself is an R-module. R acts on itself by left multiplication and so what we have is eta R from R to R. So, this should be an R-module homomorphism. No, not an R-module homomorphism, a homomorphism of Abelian groups because, well, eta R should be from omega R to omega R, a homomorphism of groups.

Now, in this R we have an element 1 and let a equal to eta R of 1, and what I want to say is that we will see that eta is completely determined by a. So, what will happen is that to each natural transformation we will be able to associate an element a of R. Well this belongs to omega R but as I said that is the same as R. okay, so let us see this, so suppose M is any R-module can we figure out what eta subscript M is based on the fact that eta R of 1 is a?

So, let us try to figure out what eta M of little m so, and let m be any element of M, we want to figure out what is eta M of little m. So, for this what we will do is let phi m phi subscript m from R to M denote the R-module homomorphism which takes r to r times m. You can check that this is actually an R-module homomorphism and now let us apply naturality to this phi m.

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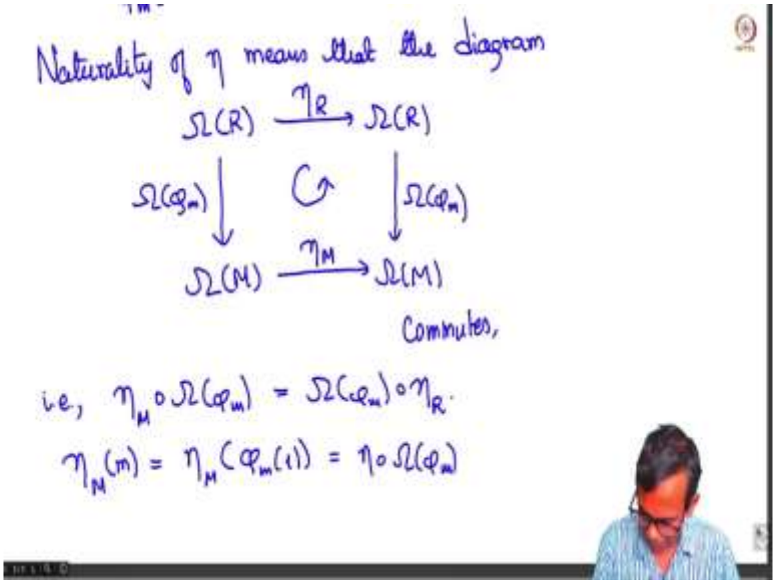


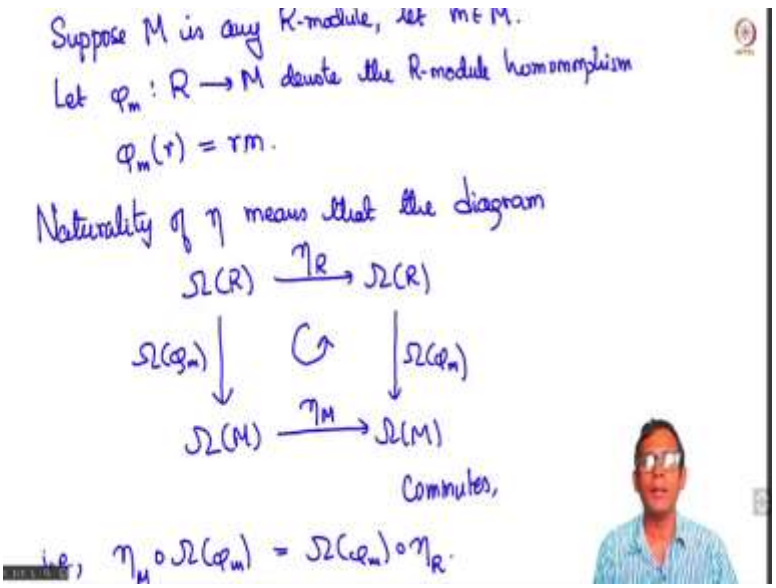


So, naturality of eta means that the diagram involving eta R and eta M so we have omega R to omega R we have here eta R and then we have omega R to omega M which is omega phi m which is just phi m as a function and here we have omega R to omega R and this is also omega phi m. This is omega R to omega M, this is also omega phi m. So, this is again phi m as a function and here we have eta M and naturality means that this diagram commutes.

So, we signify commutes by drawing a circular arrow like this. What is commuting of this diagram mean? That means that eta M circle omega phi m is omega phi m circle eta R. Now, let us see if we can evaluate eta M of m. Well eta M of m is eta M of phi m of 1 is going to be M. Let us recall the definition of phi m, phi m of r is r times m so phi m of 1 is just m, so this is eta M phi m of 1, but that is the same as eta circle omega phi m

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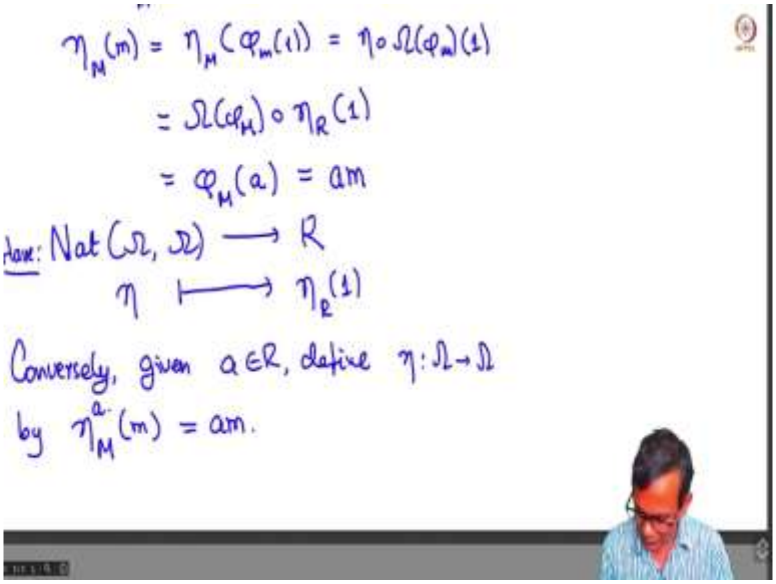


And that is the same as, by the commutativity of this diagram that is omega phi m circle eta R, sorry! Of 1, and so that is omega phi m circle eta R of 1, which is phi M of eta R of 1 which is a, which is am. So, what we see is that eta M is complete determined by this element a of r and the fact that eta is a natural transformation.

So, each natural transformation from omega to omega gives rise to an element of R. Can we do the reverse? So, what we have constructed is from Nat omega omega to R an eta goes to eta R 1. So, this is what we have, a function from endomorphisms of omega to R taking the natural transformation eta to eta R of 1.

Now let us try to compute, construct the inverse of this. So, conversely given a in R define eta from omega to omega by eta M of m is equal to am. We will call this eta superscript a so this is the eta associated to a, a natural transformation associated to a.

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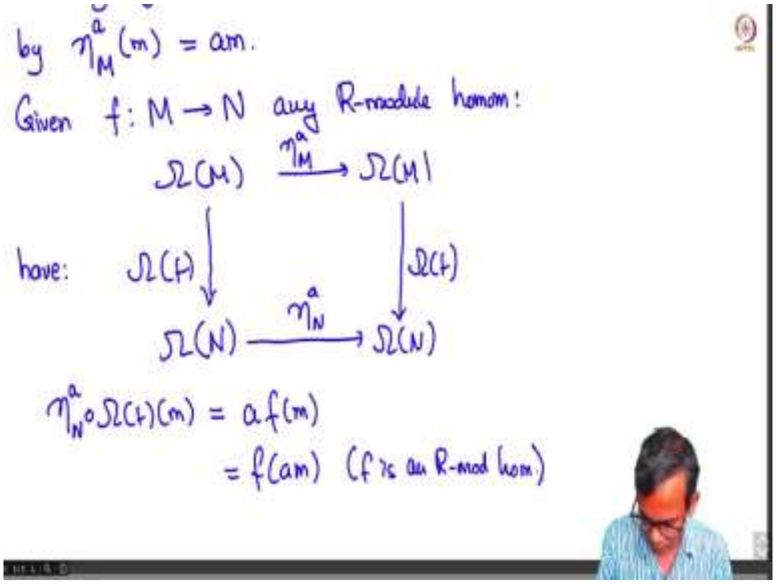


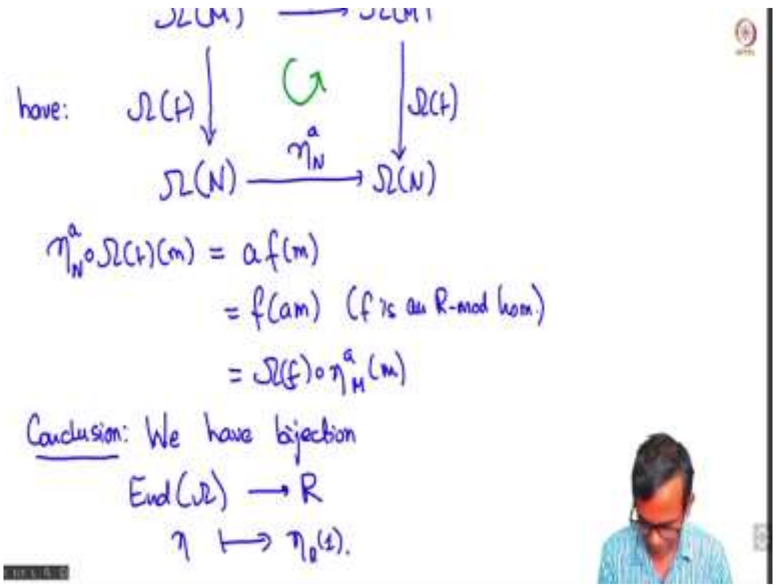


Is this a natural transformation actually? You need to check that, so given any R-module homomorphism we need to check that this is a natural transformation so we need to check if the diagram omega M to omega N here we put omega f and then we have omega M to omega N. This is now how I want to do it, I want to say omega M to omega M and here I will put eta subscript M here I will put omega f and here I will put eta subscript N, so this will go to omega N and here we have omega f.

And we want to know if this diagram commutes. So, we want to know if eta N circle, so this is eta N, a N circle omega f of m whether this is equal to omega f circle eta a of M. So, let us evaluate this thing. So, this is a of f of m right? By definition, because omega f is just f as a function and then this eta N a just multiplies it by a, but because f is in R-module homomorphism this is f of am because f is an R-module homomorphism.

And so this is omega f circle eta a M of m. So, we have that indeed this diagram does commute. So, we have constructed a bijection. I mean clearly eta a of R subscript R of 1 is a just by definition so this is the inverse of eta goes to a so we have a bijection from End omega to R given by eta goes to eta R 1.

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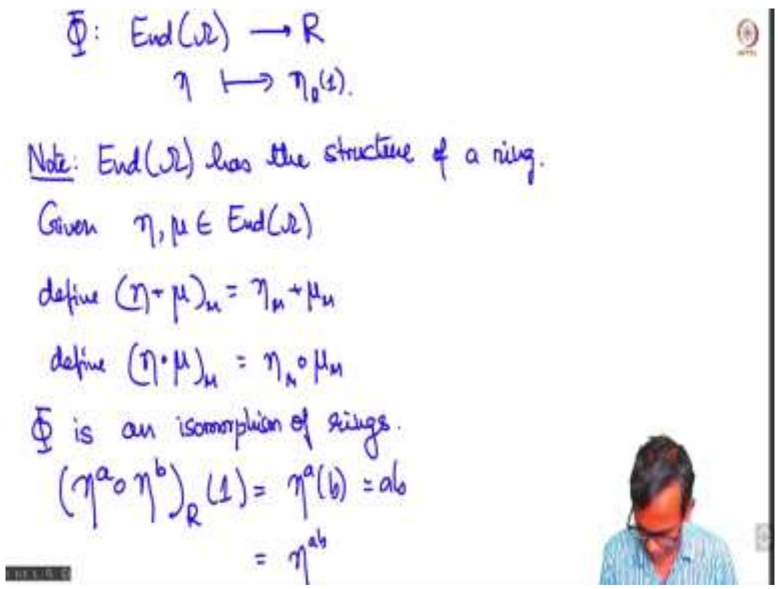

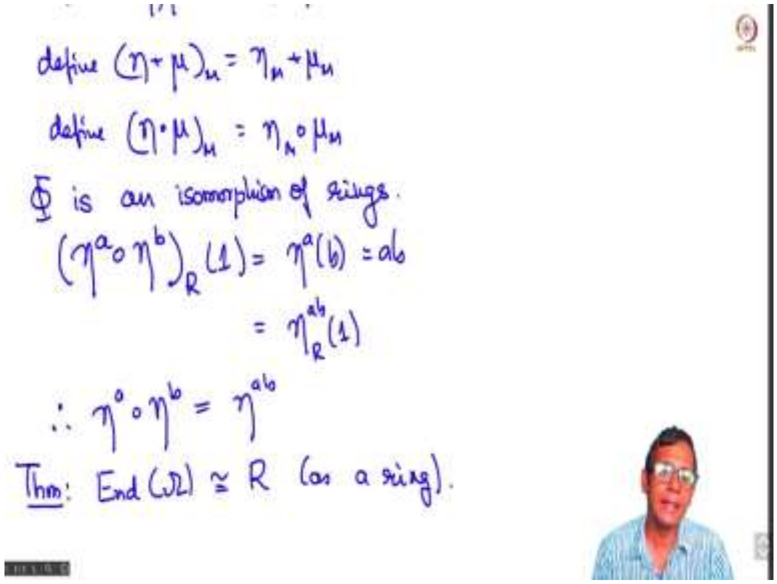


And we can also check that, so this End omega has the structure of a ring not just a monoid. How is addition given, given eta mu belonging to End omega you can talk about eta plus mu. Eta M plus mu M for every M, you can check that this will also be a natural transformation and you can define eta composed with mu of M to be eta M circle mu M.

And so, this has a structure of a ring, R also has a structure of a ring and this map is actually an isomorphism of ring so let us called this a phi. Phi is actually a ring isomorphism of rings. The main thing to check is that if you take eta superscript a circle eta superscript b then this thing at 1 is eta a of b which is ab, which is the same as eta superscript ab R of 1.

Therefore eta a circle eta b is eta ab, because this value completely determines the composition so because they agree on R at 1 they must be equal, and so finally what we have is a following theorem End omega is isomorphic to R as a ring. Now this is a remarkable observation because what it says is that okay you just start with the category of R-modules and from that category you can recover the ring R itself.

So, in some sense it is saying that the ring comes from the category so if two categories of R-modules are actually isomorphic then the underlying rings are also isomorphic.

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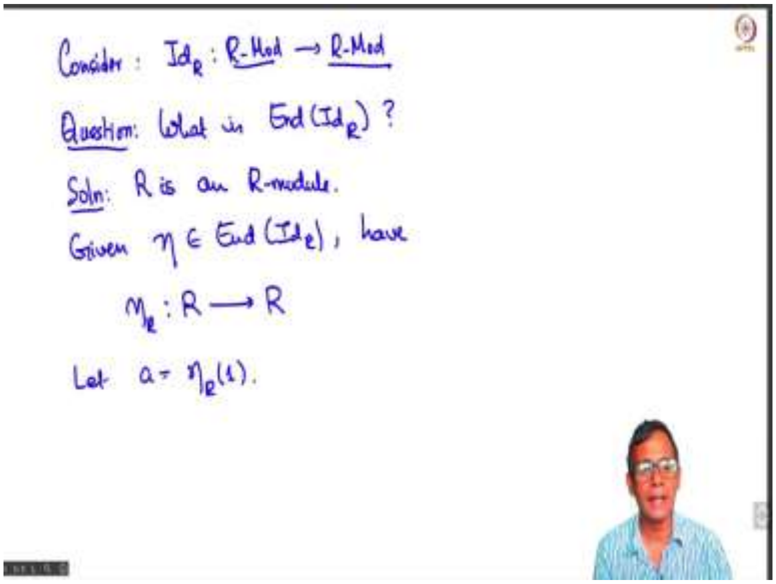


Let us look at a variant of the previous problem. Earlier we had the forgetful functor from R-modules to R-modules. Now let us consider the identity functor from R-modules to R-modules. So, this is the functor which takes each R-module to itself and each arrow of R-modules that is each R-module homomorphism also to itself.

And now the question is what is End of this identity functor? Now we can proceed as before we have an R-module R so given eta in End identity R we have eta R. Now, earlier this eta R was from omega R to omega R but this time it is R-module homomorphism from R to R and now let a equal to eta R of 1. But as before we will see that a completely determines eta R why is that?

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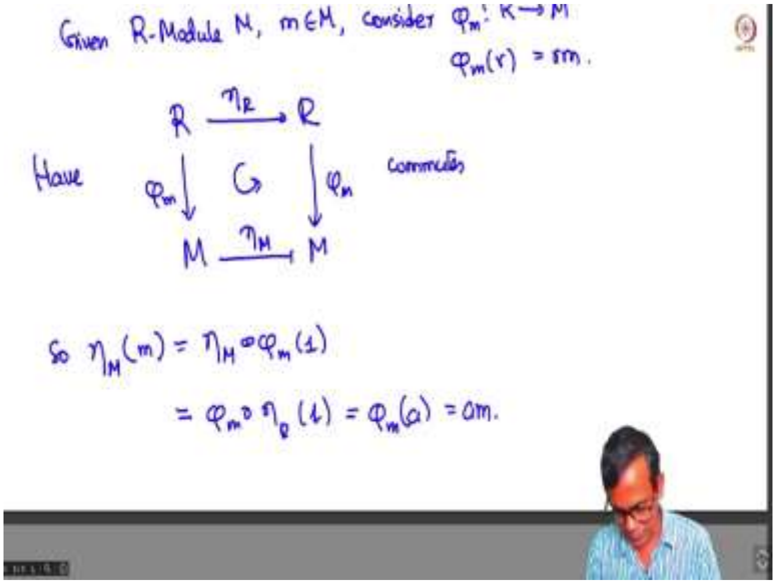


Well, just as before given any M, R-Module M and any element m in M consider the R-Module homomorphism phi m from R to M given by phi m of r equals rm, and now we have this commutative diagram R to R eta R. R to M phi m, R to M phi m and M to M eta M, this commutes.

Which means that eta M of m is eta M of phi m of 1, which is phi m of eta R of 1 by commutativity which is phi m of a which is am. So, as before we see that eta is completely determined by this element a. So, eta M of m is a.m. for every element m of every R-Module capital M. So, now let us, but what changes this time is that not every element a of R gives rise to a natural transformation.

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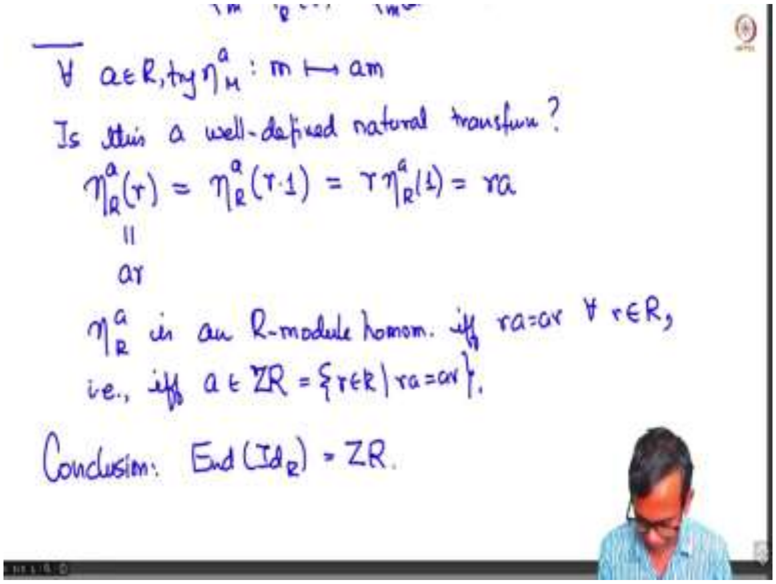


So, the key issue here is that, so what we have is that for every a in R we have eta a M must take m to am. We can try to define this but is this a natural transformation? Is this a well defined natural transformation? So, I am just trying this out okay. So, let us just trying it out on R. So, we have eta R of r, well on one hand this is going to be just ar by this calculation here eta M of m is am.

So, I am just applying r here but on the other hand this needs to be an R-module homomorphism. So, this should be eta a R. I can write this as r times 1, but that is by definition because this is an R-module homomorphism, this is r times eta a R of 1 but that is r times a. So, eta R a is an R-module homomorphism if and only if ra equals ar for every r in R.

In other words if and only if a belongs to the center of R that is just my definition the set of all r in R such that ra equals ar, the center of the ring R. It is easy to show that if a is indeed in the center of R then eta a is an endomorphism of the identity functor and so the conclusion is that End of the identity functor of R is the center of R.