Algebra-II
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Lecture 55
Functor Categories

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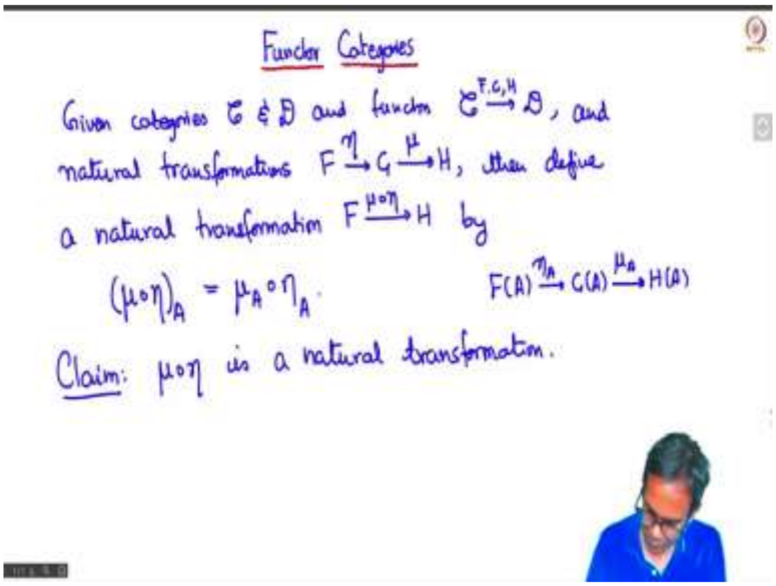

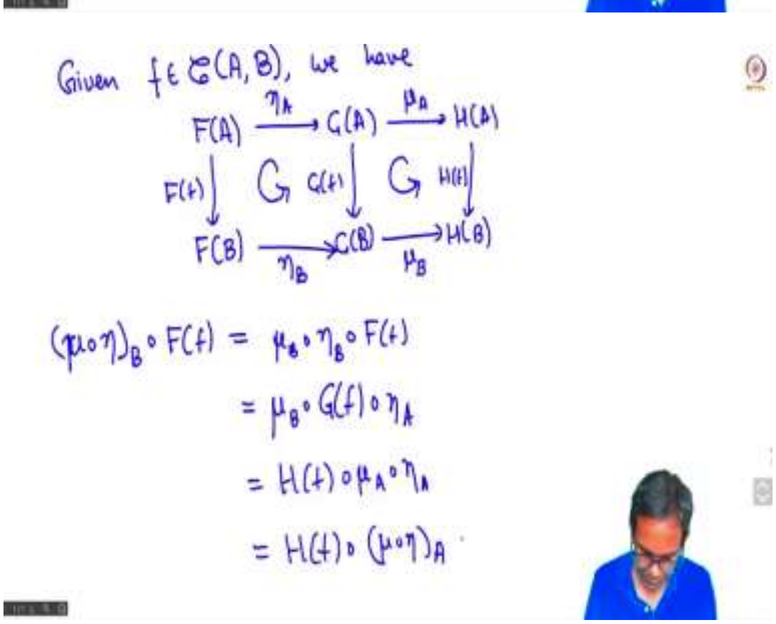

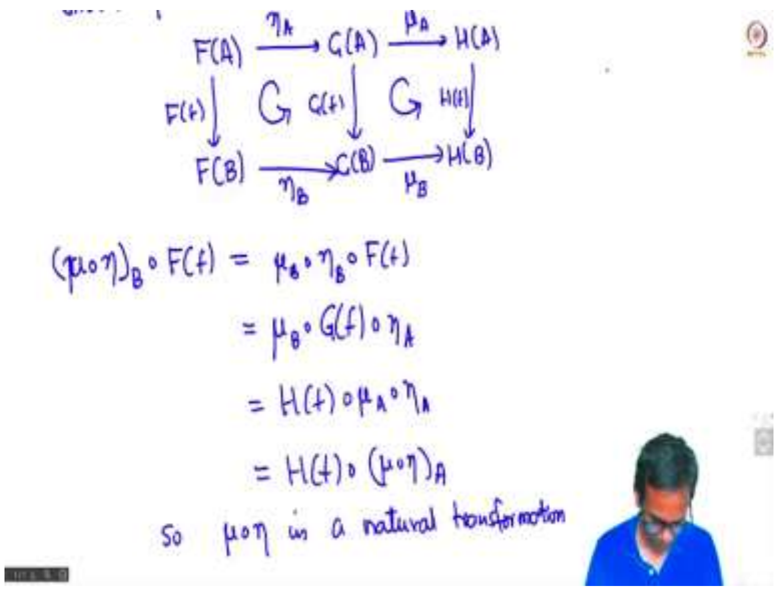

We have seen categories whose objects are categories and arrows are functors. In this lecture I am going to show you categories whose objects are functors and arrows are natural transformations. To order, in order to construct these categories we need to understand the composition of natural transformations.

So, suppose we have two categories C and D and functors, three functors say from C to D all from C to D. Let us say F G and H and natural transformations say from F to G we have a natural transformation eta and from G to H we have a natural transformation mu. Then we can form a natural transformation from F to H as the composition of mu and eta.

We will call it mu circle eta and it is defined by, we need to say what it is for every object of C, so mu circle eta of every object of A of C is mu of the object A circle eta of the object A. Recall that mu of A is going from F of A to G of A and, sorry! Eta sub A is going from F of A to G sub A G of A and from G of A to H of A we have eta mu subscript A.

And so it makes eminent sense to compose them and get mu subscript A circle eta subscript A. And we need to check that this is a natural transformation. So, how do we check that, so claim is that mu circle eta is a natural transformation. We will use the fact that mu is a natural transformation and eta is a natural transformation.

So, given f an arrow in the category C from A to B we have, so we have F of A to G of A we have eta A and from G of A to H of A we have mu A. And similarly from F of A to, F of B to G of B we have eta B and from G of B to H of B we have mu B and vertically we have the arrows F of f, G of f and H of f.

Now, the fact, that eta and mu are natural transformations means that this square here commutes and this square here commutes and we can put them together to show that this big rectangle here commutes. So, now what we need to show is that mu B circle eta B circle, let us just look at mu circle eta B circle F f. So, by definition mu circle eta B is mu B circle eta B circle F f, but eta B circle F f is equal to G f circle eta A, you can go this way or you can go this way in the square on the left.

So, this is mu B circle G f circle eta A. And now what you can do is you can use the fact that the square on the right commutes, so mu B circle G f is H f circle mu A. So, this is H f circle mu A circle eta A, but that is the same as H f circle mu circle eta of A. And this is establishes that mu circle eta is in fact a natural transformation.

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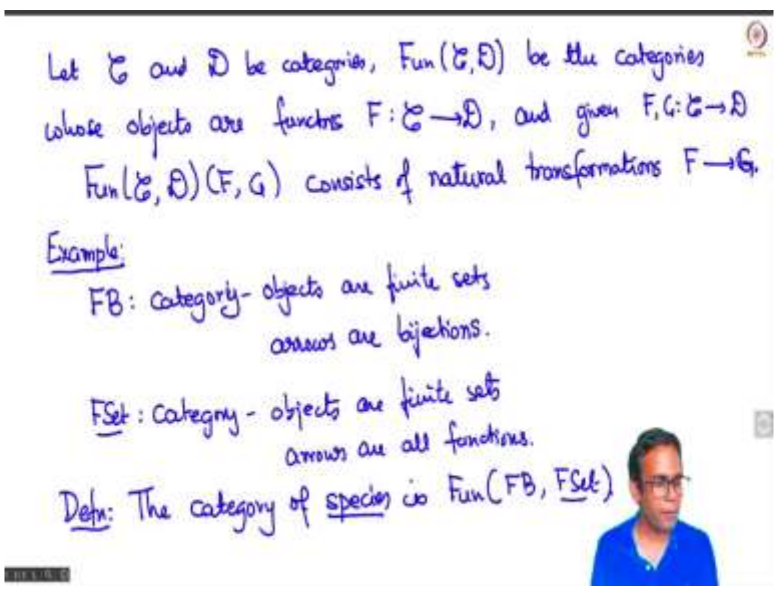

We are now ready to define a functor category. So, let C and D be categories and let Fun C D be the category whose objects are functors F from C to D and in Fun C D the arrows between two functors and given functors from C to D, the arrows in Fun C D from F to G consists of all natural transformations from F to G.

So, let us, so this you may see, think as building up several layers of abstraction, but what is it good for? Such categories are used a lot nowadays in algebra. Let me give you a famous example which is the category of species. So, this was developed by the mathematician Andre Joyal in 1980 and he used it to solve combinatorial problems. I will say more about that later, but let us construct the example of species, the category of species is a functor category.

So, firstly I need to define some categories, so FB is the category whose objects are finite sets and arrows are bijections. So, you do not allow any function from one finite set to another you only allow bijections from one finite set to another. So, you have no arrows between finite sets which have different cardinality of course you still have an identity arrow from every finite set to itself which is a bijection.

So, this is a category and the other category I will take is FSet this is the category of finite sets with functions are, just all functions. Arbitrary functions are arrows in this category. Then definition is the category of species is the category of functors, is the functor category of functors from FB to FSet.

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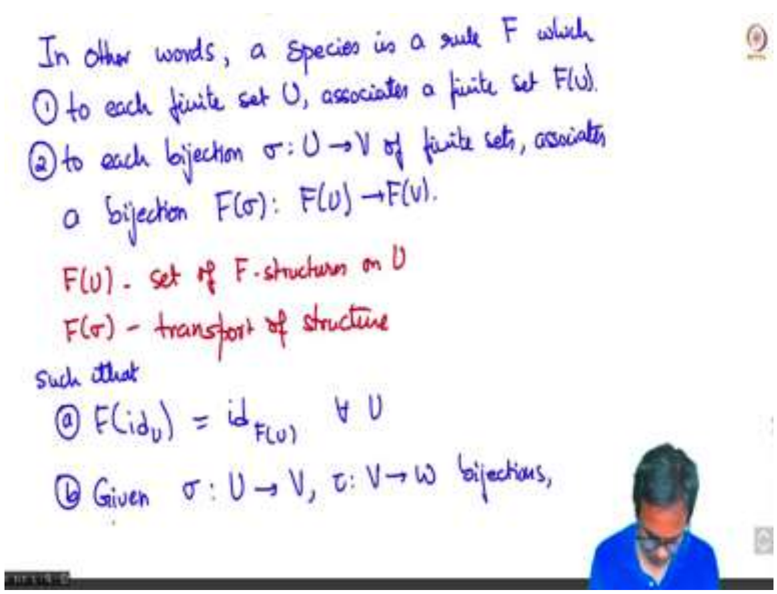

So this succinct definition keeps track of a lot of information. So, let me translate this into planer English, so in other words an object in this category which is called a species is a rule F. So, it is going to be a functor, what does it do? Firstly to each finite set it must associate another finite set, to each finite set U associates a finite set which we will call F of U.

That is at the level of objects and the other thing is to each bijection, let us say sigma U to V of finite sets associates a bijection F sigma from F U to F V. See a priori when we say F is a functor, then F would only associate a function from F U to F V, but since sigma is an isomorphism F sigma will also be an isomorphism.

The inverse of F sigma will be F of the inverse of sigma and so F sigma will actually be a bijection and what we should think of this F as a way of putting a certain structure on U. So, the set F of U will be all the different ways of putting a certain structure on U. And this F sigma will be called transport of structure. So, in, I will give you examples where it will be clear about what this means.

So, this will be, F U will be the set of F structures on U and F sigma is called transport of structure. We are not done yet with unwinding the definition. These sets and these transport functions must satisfy certain axioms. So, the axioms are, axioms sort of functor namely that F of identity of U is the identity function of F U for every finite set U and the second axiom is that given bijections.

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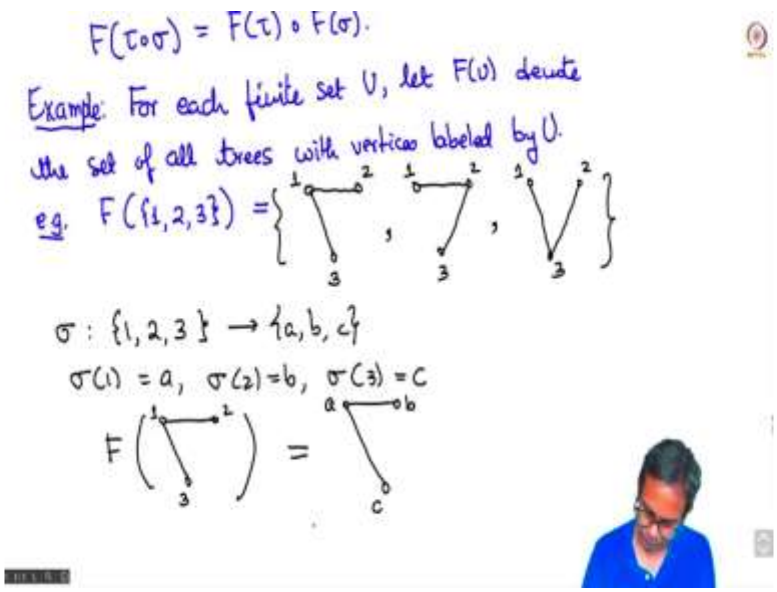

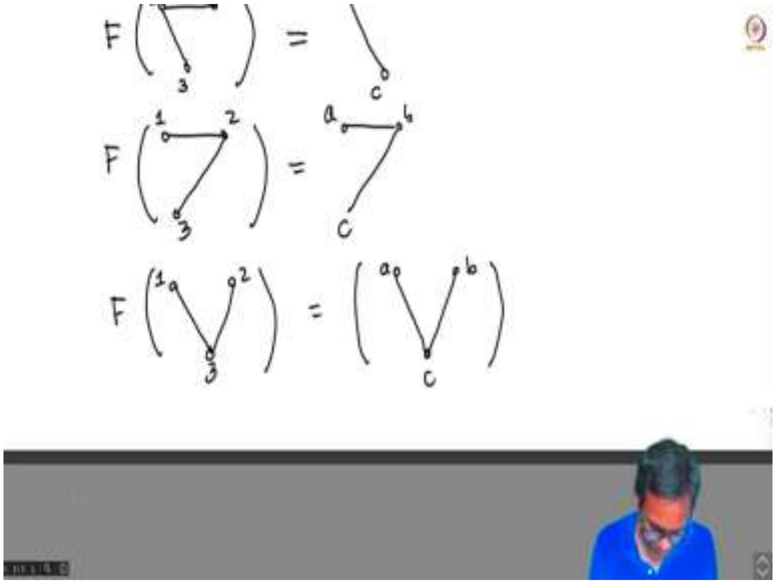

F of tau circle sigma needs to be equal to F of tau circle F of sigma. Let me give you an example of this, of species. So, this is historically interesting example for each finite set U let F U denote the set of all trees. So, this is from graph theory. A tree is a graph which has no cycles; it is a connected graph with no cycles. So, set of all trees with vertices labeled by U.

Let me, so this is the structure we are talking about is a tree structure. So, F of U is the set of all tree structures on the set U. So, if we take U equals, so an example inside this example is that F of 1 2 3. So, now we have three points 1 2 3 and we want to find all possible ways of constructing a tree with these three points as a vertex.

So, there are actually, it is not difficult to see that there are only three such trees 1 is where there is no edge between 2 and 3, another is where there is no edge between 3 and 1 and another one is where there is no edge between 1 and 2. This is the image of the set 1 2 3 on the species for the species of label trees. And now suppose we have sigma.

Let us take sigma to be a function from bijection from 1 2 3 to a b c another set where sigma of 1 is equal to a, sigma of 2 is equal to b, sigma of 3 is equal to c, then each of these tree structures on the set 1 2, trees with labels 1 2 3 will give rise to tree with labels a b c where you will be replacing the labels of the original tree with the transformed labels. So, we will have that F of 1 2 3 is equal to a b c. So, this is F on the level of morphisms.

And similarly you can do F of 1 2 3 would be the tree a b c and F of, so the structure on the set 1 2 3, the tree structure on the set 1 2 3 is transported to a tree structure on the set a b c using the bijection sigma. Now this might seem all very simple the definition is just a line once you understand what functor category is.

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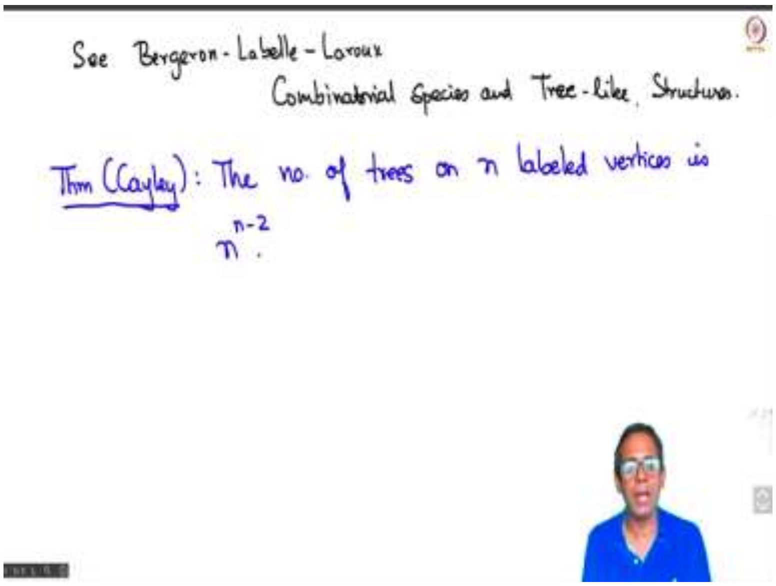

And however it turns out to be very useful and there is a nice reference for applications of the theory of species, is the book by three authors Bergeron, Labelle, and Leroux and it is called Combinatorial Species and Tree-Like Structures. To give you an example of the power of this theory let me mention that Andre Joyal in his original paper where he introduced species he used this theory to give a very, very beautiful bijective proof of a theorem of Cayley.

So, here is the theorem of Cayley. The number of trees on n labeled vertices is n to the power n minus 2 and Joyal used the theory of species to come up with a very, in a very natural way with the bijective proof of this theorem. So, you can look at this book or you can also look at the famous by Aignar called Proofs from the Book where Joyal's bijection is given but without any mention of species.