Algebra-II
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Lecture 54
Natural Transformations

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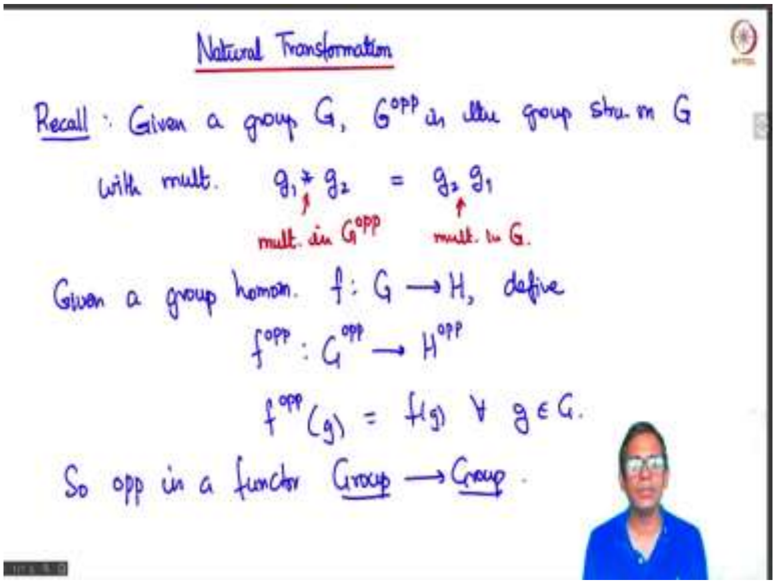

There are three fundamental concepts in category theory; the first is the definition of a category itself. The second is the definition of a functor and so functors are things which relate to different categories. The last and final fundamental definition in category theory is that of a natural transformation, which relates to different functors.

Let me start with an example. So, recall that we have a group G, G opp is the group with underlying set G. So, it is a group structure on G with order of multiplication reversed with multiplication defined by, so g 1 star g 2 so this star denotes the multiplication in G opp, and that is equal to g 2 g 1 where this denotes multiplication in G itself.

And you can check that this actually is a well defined group, it will satisfy identity associated with the inverse and so on. And, in fact, this opp is more than just a way of associating a new group to an old group, it is actually a functor. So, it turns out that opp is a functor. You can also define it on homomorphisms.

Given a group homomorphism f from G to H we can define f opp from G opp to H opp by just defining f opp of g is equal to f of g for all g in G, and you can check that f opp is a group homomorphism. I will leave that as an exercise to you. And so opp defines a functor from the category of Groups to the category of Groups.

So, this the category whose objects are groups and arrows are group homomorphisms. Now, there is a theorem in group theory which says that every group is isomorphic to its opposite.

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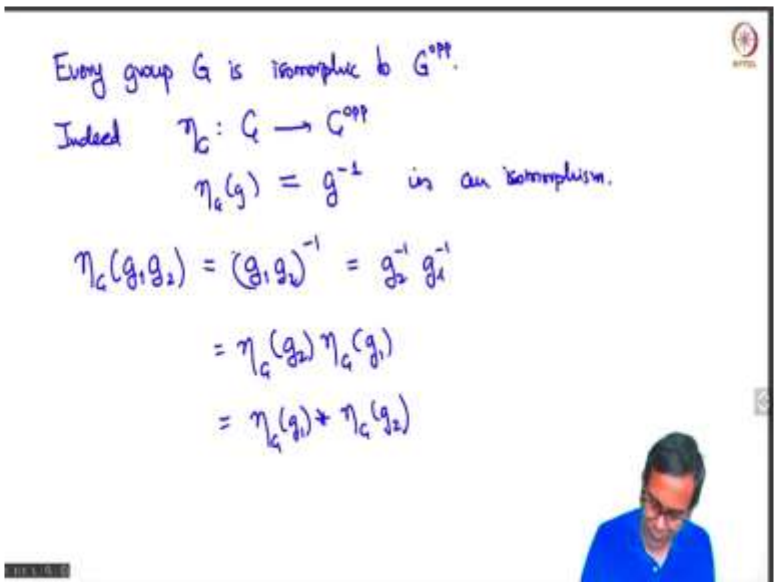

And how do you construct this? Define eta G from G to G opp by, it is not, you cannot take G goes to G, you have to take eta G of g is equal to g inverse, is an isomorphism. We just need to check eta G is clearly bijection, we just need to check that it is a group homomorphism. Now it clearly takes the identity to identity. Let us just check that it preserves multiplication.

So, eta G of g 1 g 2 so this is multiplication in the group g where this is going to be g 1 g 2 inverse which is g 2 inverse g 1 inverse which is eta G g 2 eta G g 1, which is, this is not quite multiplication in G opp, this is still multiplication in G. To make it multiplication in G opp we need to write g 1 star eta G g 2, we reverse the order and so you see that eta G of g 1 g 2 is eta G g1 star eta G g 2.

So, this is group, is an isomorphism of groups, but there is something very spacial about this family of isomorphism. So, for each group g we have an isomorphism eta G, so this eta G is actually a family of isomorphisms, one for each group. There is something very nice about this family of isomorphisms and let me explain that in a minute.

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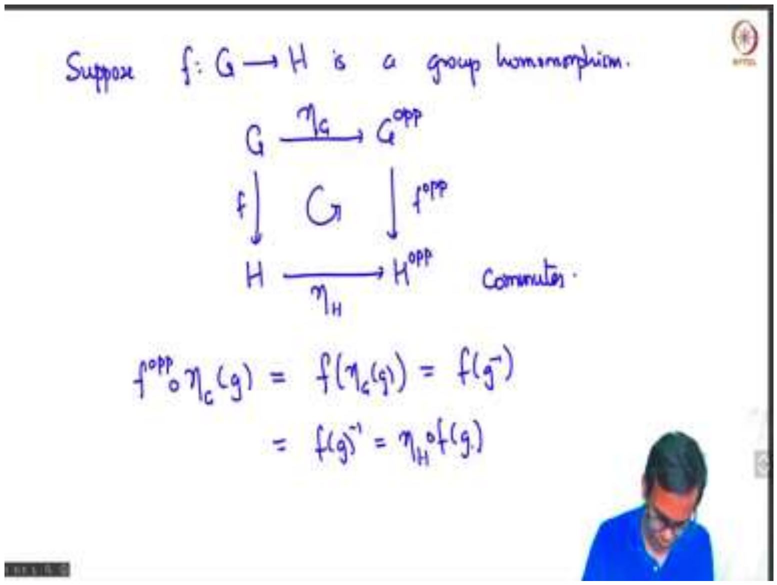

So, suppose you have a group homomorphism. So, then you can draw this diagram here, you have G to G opp and you have eta G, you have H to H opp and you have eta H. Now from G to H you have f and from G opp to H opp you have F opp and this diagram commutes, in other words f opp circle eta G of g well that is equal to just f of eta G of g because f opp is justified as f so that is f of g inverse.

But f of g inverse is f of g inverse because f is a group homomorphism and so that is eta H of f of g, therefore this diagram commutes and the commutativity of this diagram for every arrow in the category of groups is basically the definition of naturality. So, a natural transformation. I will give you the formal definition in a minute is a collection of arrows in the target category so let me give you the formal definition now.

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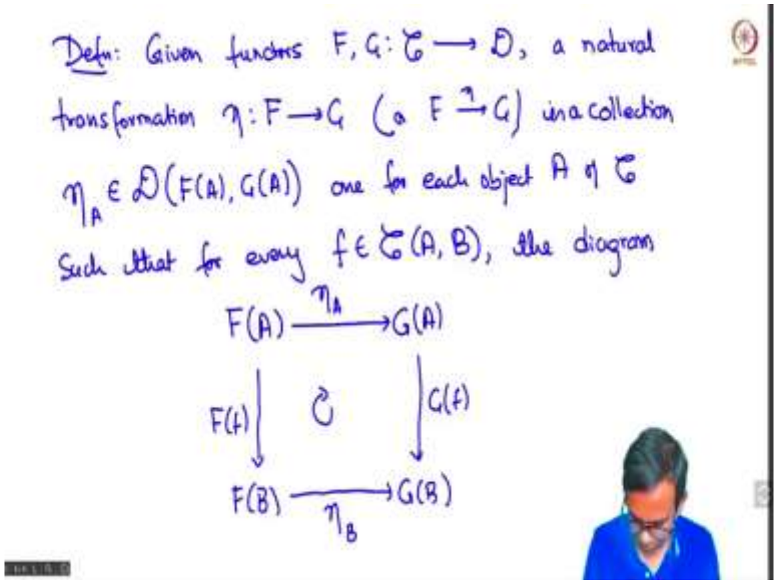

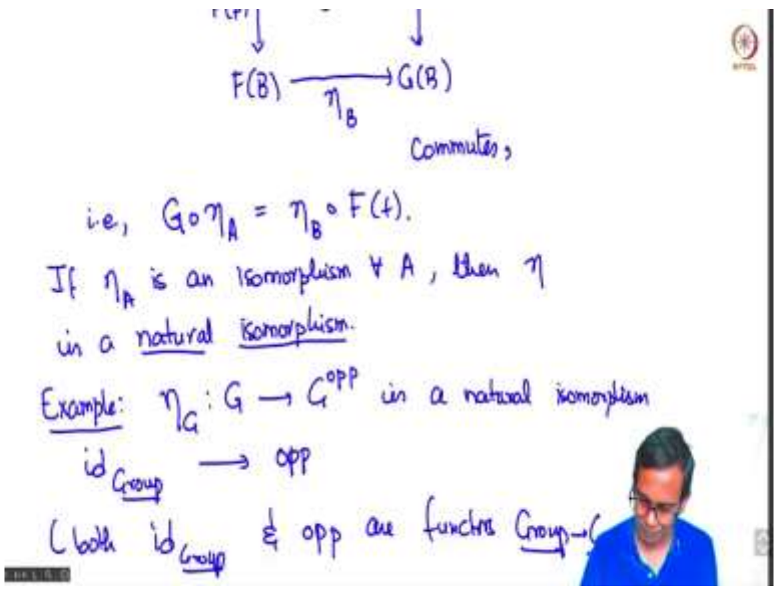

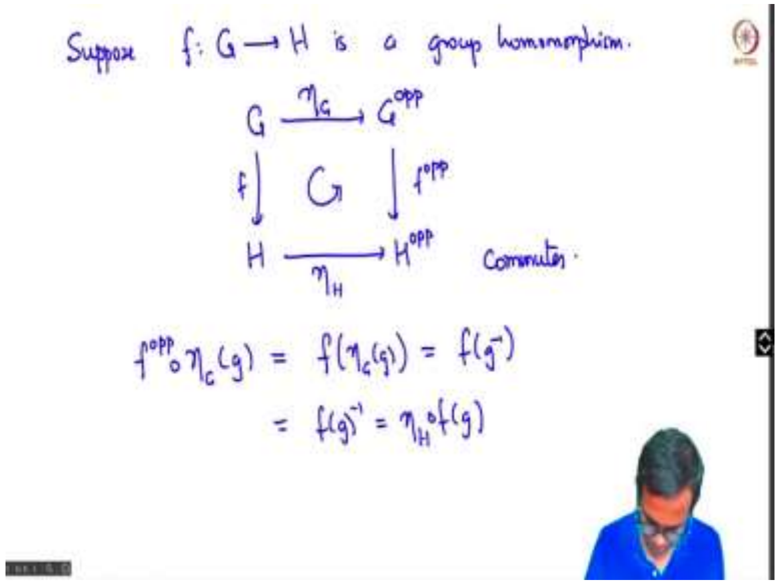

So, given two functors F and G from a category C to a category D a natural transformation and here the notation is eta goes from F to G or sometimes we will write F eta G so eta is an arrow in some category which we will come to later, but eta is of natural transformation from F to G is a collection eta G, eta let us say A and this eta A has to be in the category D and it should be an arrow from F A to G A.

The collection of arrows such that one for each object A of C such that the diagram, maybe I should say such that for every arrow f in the category C between objects A and B the diagram. So, we draw the analog of this, more general version of this diagram so we have, firstly we have F of A and then we have G of A and here we have the arrow eta A.

This is an arrow in the category D, but then we also have because of F we have an arrow F f and this is to F of B and we have an arrow G f this is to G of B. This is again, these are arrows in the category D and then we have eta B. This diagram commutes, i. e. G f circle eta A is equal to eta B circle F f, and in particular if eta A is an isomorphism for every object A then we say that eta is a natural isomorphism.

So, natural isomorphism is a special kind of natural transformation which is made up of isomorphisms. So, now just get everything clear so given to functors F and G a natural transformation from the functor F to the functor G is a collection of arrows in the target category D and it goes from F of A to G of A and this diagram must commute. The upshot of this definition and the previous example is that eta G which takes G to G opp, but we can think of this G as the identity functor applied to the group G.

So, we have from the category of groups to the category of groups. We have the identity functor which takes each group to itself and each arrow to itself. So, this is a natural isomorphism between the identity functor from groups to groups to the opposite functor, from groups to groups, both identity Group and opposite are functors from Group to Group.

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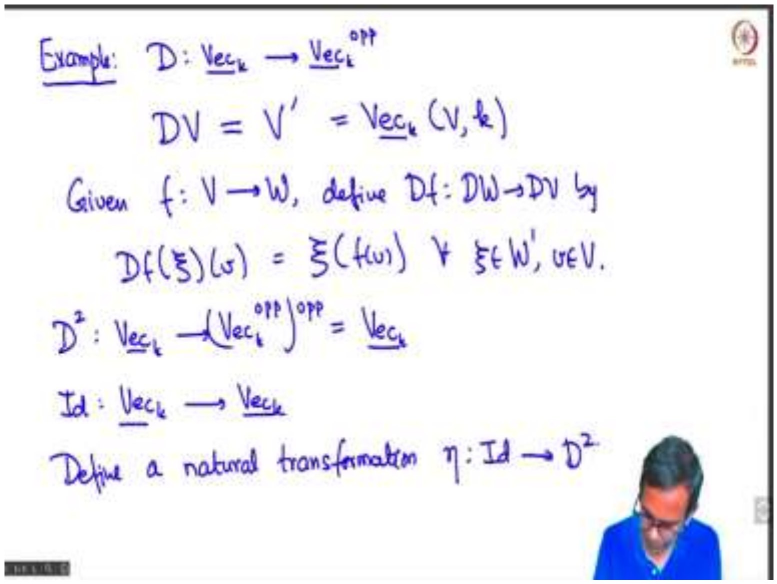

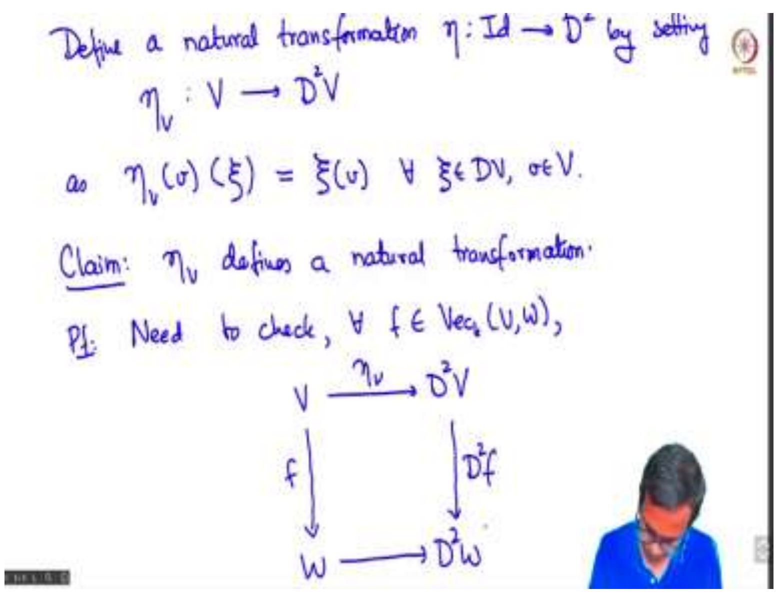

Let us look at some more examples. So, recall that we have this functor from vector spaces to the opposite category of vector space or in the other words a contra-variant functor from the category of vector spaces to the category of vector spaces where for every object V we define DV to be the linear dual so this is the category theory language, you can say this is the linear maps from V to k.

And now you can also talk about, this is a functor given f from V to W, we can define Df from DW to DV by Df of xi of v equals xi of f of v for every xi belongs to W prime and v belongs to V. Now this functor can be squared. So, we can compose D with itself. So, we have D squared is a functor from Vec k to Vec k opp opp, but double opposition is just Vec k itself, because if you reverse all the arrows two times you are back to the original arrow.

So, D squared is a functor from vector spaces over k to vector spaces of k and it takes each vector space to its double dual. And now we also have identity functor from vector k to vector spaces Vec k, and this just takes each vector space to itself, and now I will define a natural transformation from the identity functor to the double dual functor. So, eta is going from the identity functor to D squared.

And so what I need to do is for every object V I need to define eta subscript V, so eta V and so this should go from V to D square V. So, eta V takes a vector v and it gives us a linear functional on DV so this should, so suppose xi is in DV then it should give me element of the field K and this can be just defined to be xi which is a linear functional on v evaluated at v so this is for every xi in DV and v in V.

So, this formula is, does not really, it is a very nice formula, it does not really involve what is V and what is xi and so, it is just a formal kind of nice way to put together things and because of that it ends up satisfying naturality. So, claim is that this collection eta V defines a natural transformation. So, what do we need to do, we need to check something.

So, we need to check a certain diagram commutes so whenever we have that for every f in Vec k from V to W the diagram we have from V to D squared V we have eta V, we have W to D squared W, we have D squared f and we have eta W from W to D square W.

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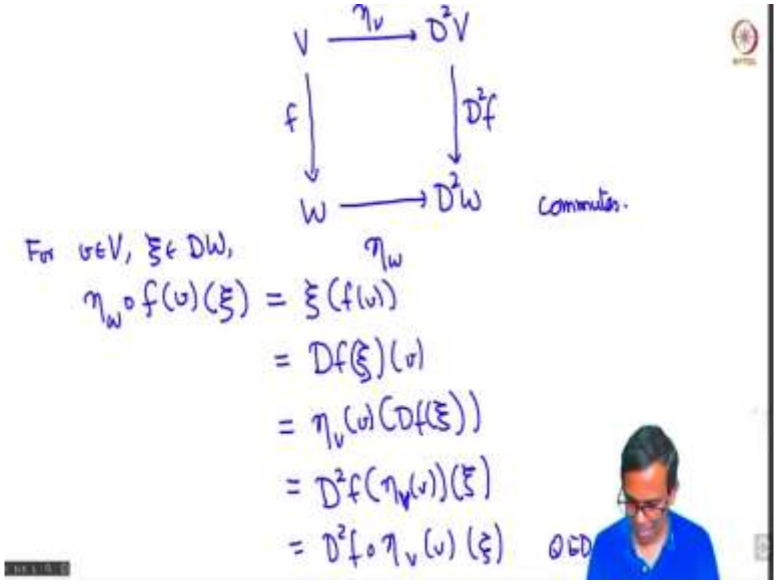

We need to check that this diagram commutes. Okay so let us do that, one the one hand we have eta W circle f and this takes a vector v and so let us evaluate that so that needs to be evaluated so what do you get when you do that, you get linear functional on DW so we have to take an element of DW and we have to return element of the field k.

So, this is going to be by definition of eta W this is going to be xi evaluated at f v, after all f v is an element of W so this linear functional. So, here we have for every v in V and xi in DW so this is xi of f, that is the definition of eta What, and that is the same as Df of xi evaluated at v, that is from the definition of D, and that is the same as eta V of v evaluated at Df of xi, this is from the definition of eta V and so, but that is the same as.

So, if you look at the definition of D again this is the same as D square f evaluated at eta V v evaluated at xi. So, what we get is that this is just D squared f circle eta V v evaluated at xi. What we get is that D squared f circle eta V is equal to eta W circle f, so that completes the proof of the naturality of eta. One last example of a natural transformation which is of a very different nature from the two we have considered so far.

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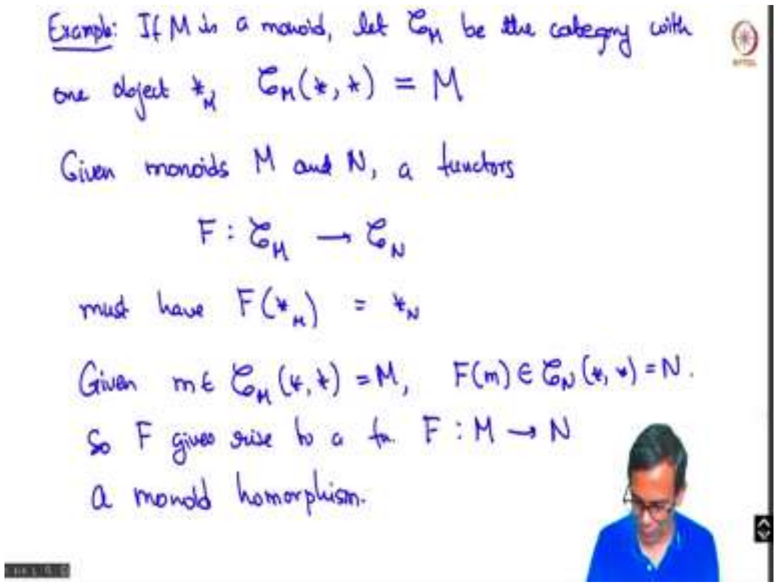

Let us consider the categories with one object. So, remember that if M and N are monoids if M is a monoid there is a category C M, let C M be the category with one object star and C M the arrows from start to star in C M are just given by the elements of the monoid M, and now this monoid M has an identity element that is going to be the identity arrow and the composition of arrows will just be given by the multiplication operation of the monoid M.

And so now we can ask what are the functors from a category C M to a category C M. so recall, so a functor given two different monoids M and N a functor let us say F from C M to C N well there is not much it can do at the level of objects, must have F of star, this is the star of M, must be equal to the star of N. So, let us just give this, let us call this star of N, but then what is more interesting is what it does on the arrows of M, and the function F, so given M in M, so this is C M star start which is equal to M, F m is element of C N star star which is N.

So, F gives rise to a function which we will also denote by F from M to N and the functoriality of F implies that this function capital F is a homomorphism of monoids, a monoid homomorphism, which means that it preserves the product structure for monoids and takes the identity to the identity. So, these are the functors from C M to C N are just the monoid homomorphisms from M to N.

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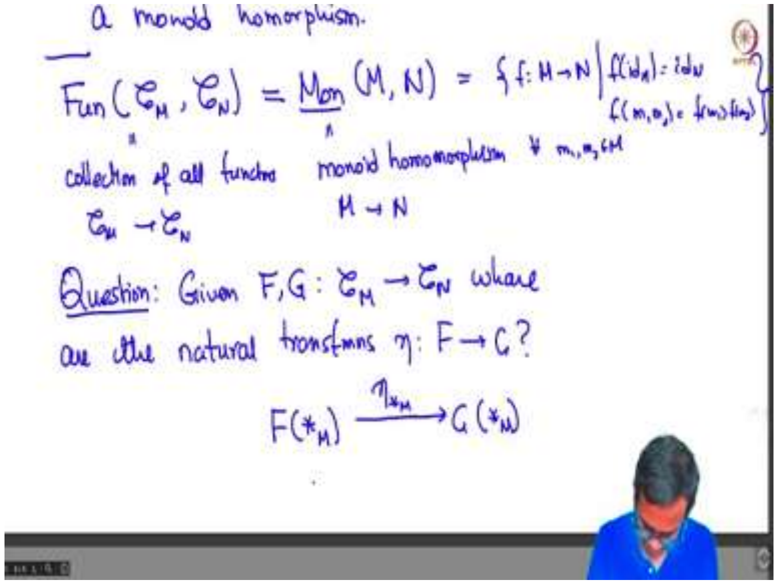

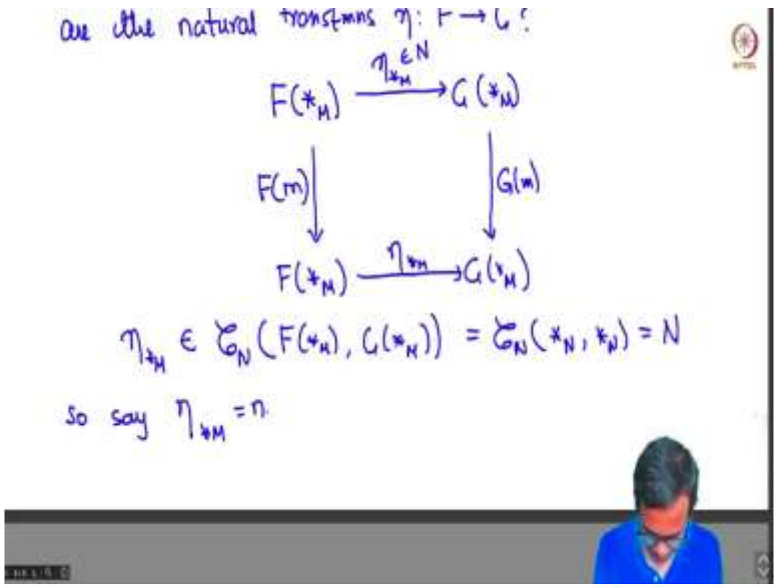

So, let me write that schematically. So, Fun C M to C N this is the collection of all functors C M to C N is equal to Mon M N where Mon is arrows in the category of monoids, which means that this consists of all functions f from M to N such that f of, the identity of M is equal to the identity of N and f of M 1 M 2 is F M 1 f M 2 for all M 1 M 2 in M.

So, that is the category of, that is the set of functors from M to N. So, now let us ask given to functors F and G from M to N, what are the natural transformations, the question, what are, so given let us say F and G functors from M to N, C M to C N what are the natural transformations eta from F to G. So, let us see what it means to be a natural transformation.

So, we have, must have the following diagram so we have F of, there is only object in the category F so we have F of M and we have F of star of, G of star of M and we must have an object eta of star of M. We must have an arrow and this would be an arrow in C N. So, this would be, we will decipher it later, let us just write down the definition of natural transformation.

So, then we have for every M we have F m here and here we have G m, G of star of M. This is eta star of m. So, we only have to find one, there is only one object in the category C M so we only have to define eta on that object. So, let us just unravel where these things live so eta star M where does this live?

This belongs in the category C N, it is an arrow from F star M to G start M, but F star M and G star M both have to be in N. So, in fact they have to be star N and start N. So, this is just N. So, eta star M is an element of M. This is in N and let us say it is, so let us say it is equal to N so say eta star M is equal to n.

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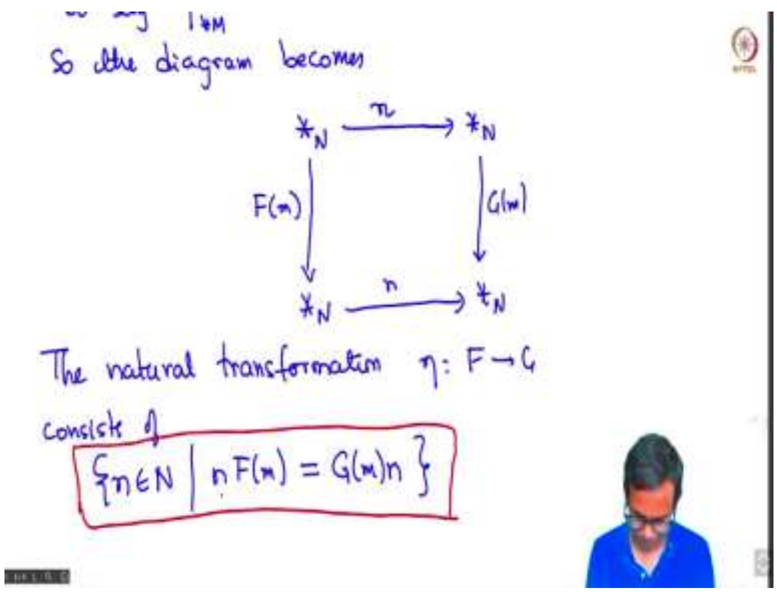

What about F of m and G of m well those are just monoid homomorphisms so let us rewrite this thing here sort of unraveling all these definitions so F of star M is star N and we have star N and we have this element n which completely determines and it is determined by eta and here we have F m which is also an element of N we have G m which is also an element of N and here again we have star N, star N and we have n.

So, the set of natural transformations from F to G consists of the of n in N such that n multiplied by F m this is a multiplication in the monoid n is equal to G m multiplied by n. so this is the collection of all natural transformations from F to G in this setting, and not all these natural transformations are going to be isomorphisms. The natural transformation n is going to be an isomorphism if and only if n is invertible in the monoid n.

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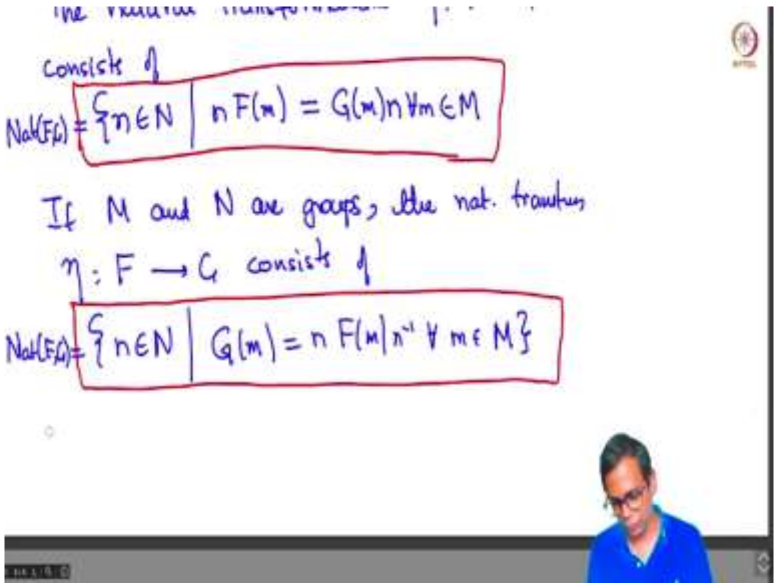

So, yeah, and if I you take the special case where M and N are groups this can be rewritten as, so let us just say the natural transformations eta F to G consists of those elements n in the group G in the group N such that we can rewrite this rather nicely as G of m is equal to n F of m n inverse for all m in M. I should say here for all m in M in fact. So, if we use the notation Nat, we can call this Nat F G. the set of all, collection of all natural transformations from F to G.