Algebra-II
Professor Amritanshu Prasad
Department of Mathematics
The Institute of Mathematical Sciences
Lecture 53
The Category of Categories

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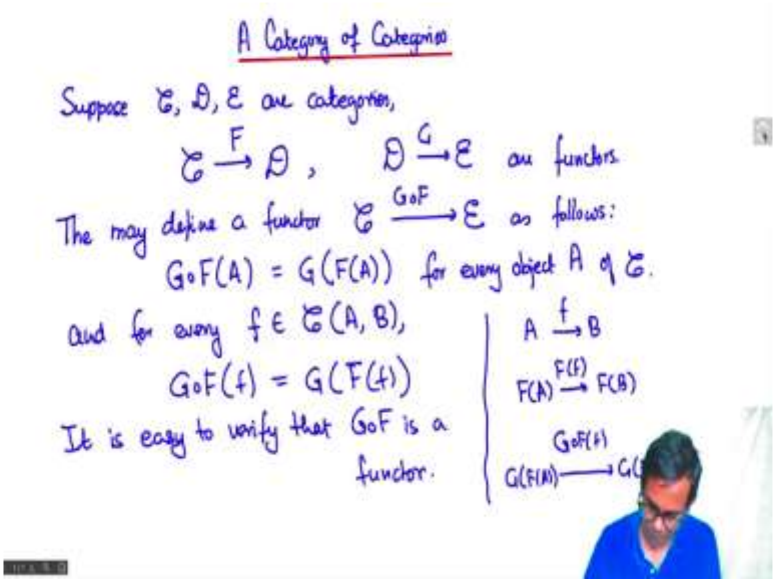

In this lecture we will see how we can from a category whose objects are categories themselves. So, to begin with let us understand the notion of composition of functors. So, suppose we have category C D and E and we have functors C to D we have a functor F and D to E we have a functor G. Then we can define a functor from C to E and I will call that G circle F as follows.

So, remember to define a functor you need to define it at the level of objects and at the level of arrows. So, at the level of objects it is going to be G circle F of A is G of F of A. This makes perfect sense because F of A is an object in the category D and then G is a functor from D to E, so this is an object in the category E.

So, G circle F of A is an object in the category E, and so this is for every object A of C and on the level of morphisms we can define it similarly for every arrow f in the category C from A to B we can define G circle F of f to be G of F of f. So, notice here that F of f is from, so f is from A to B and then F of f is from F A to F B.

This is how functors work, and then G circle F of f is from G of F of A to G of F of B. And this is an arrow in the category E. So, this defines the composition of functors, it is easy to verify that in fact G circle F is a functor. You need to check that respects identity arrows and composition of arrows. If you want, you can try to do that. So, you can compose functors and so this suggests forming a category of categories.

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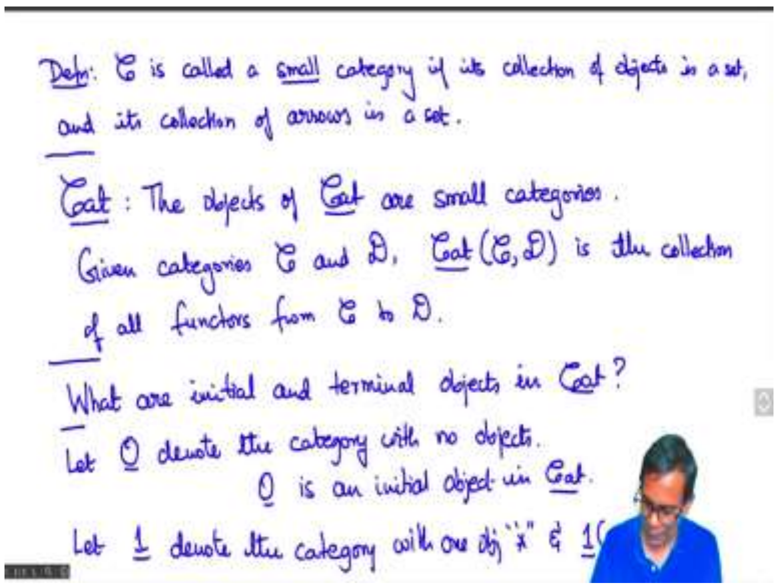

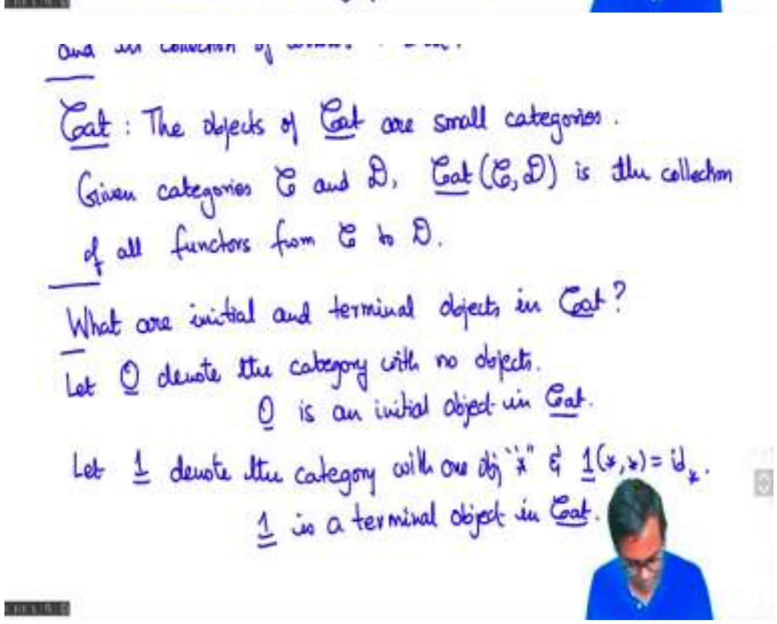

So, we will say that C is a small category. If its collection of objects is a set and its collection of arrows is also a set. If you are not interested in set theory, you can ignore this definition entirely, but this is needed in order to prevent issues like Russell's paradox from cropping up. So, now I will define The Category of Categories, so Cat this is the category whose objects are categories, are small categories.

And the arrows, so given categories C and D the arrows from C to D is the collection of all functors from C to D and as we have observed there is an identity functor from each category to itself and also we can compose functors. So, you can check that this Cat satisfies the axioms for being a category. And this thing about small categories you need not worry about it too much for most practical purposes we can replace our large category by an equivalent small category by making some technical restrictions.

So, we will not worry about these issues with small categories and Russell's paradox, I will gloss over them completely in this course but you can look at McLean's book for more details. So, let us just play with this category a little bit to become familiar with it. So, here is a question, what is an initial object in Cat and what is a final, terminal object in Cat?

You can pause the video now and take a minute to try to think what the answers are; maybe the category of sets can give you some motivation. But here is the answer so there is a category, which I will denote by 0 maybe an underlined 0, denote the category with no objects and hence no arrows either because there are no objects and this is an initial object in Cat.

Indeed given any category C there is only one functor from 0 to that category C. The functor does not have to do anything, you can think of it as the empty functor, it does not associate. There are no objects in O, so you do not need to associate to any, so there is nothing to associate and that is the only functor there is and there is another category which I will denote by 1 with one object.

Let us call it star and it has only 1 arrow namely the mandatory arrow the identity of star and this turns out to be a terminal object in Cat. Given any category C you can define an arrow to this category 1 by just taking every object in your category C to this object star of 1 and every arrow of your category C to the arrow identity star, just check, think about it and make sure that this makes sense to you. And we can also do other universal constructions in The Category of Categories.

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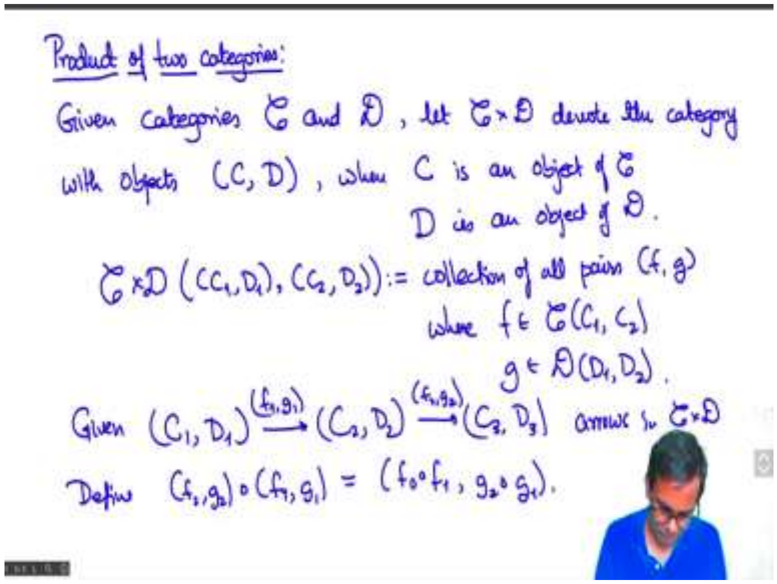

Let me give an example, let us talk about the product of two categories. So, given categories C and D we will form a, construct a category called C cross D. Let C cross D denote the category, so I need to specify what are the objects, what are the arrows and what is composition with objects of the form C comma D where C is an object of C and D is an object of D, and what are the arrows?

So, in the category C cross D let us take two objects, so let us say C 1 comma D 1 and let us take another object C 2 comma D 2. So, here C 1 and C 2 are objects in the category C and D 1 and D 2 are objects in the category D. Let us write this neatly. And this is defined to be just pairs of arrows. The collection of all pairs f comma g where f is an arrow in C from C 1 to C 2 and g is an arrow in D from D 1 to D 2 and then given objects, given arrows.

So, we suppose we have C 1 D 1 to C 2 D 2 and then to C 3 D 3 and here we have say f 1 g 1 and f 2 g 2. So, given such arrows in C cross D the composition is defined by f 2 g 2 circle f 1 g 1 is component wise composition f 2 circle f 1 g 2 circle g 1. So, this defines the category C cross D and I claim that this category is a product in the category of categories.

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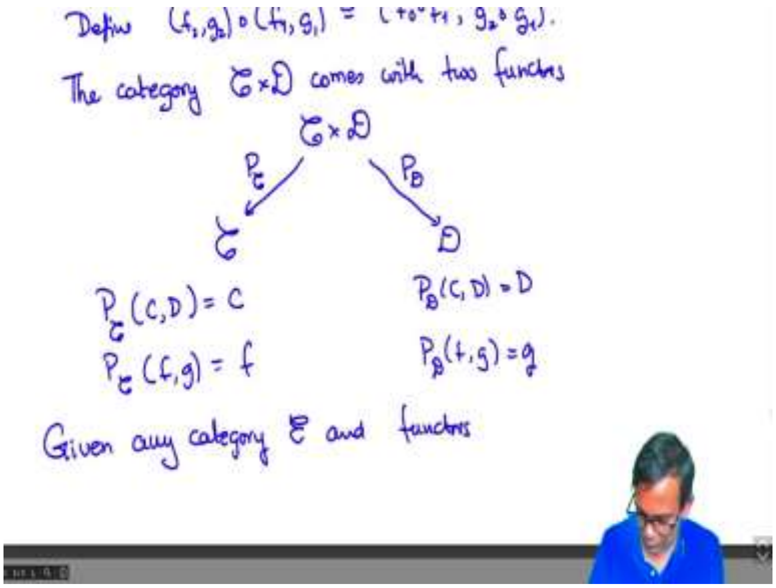

So, this C cross D comes with two functors, C cross D has a functor to the category C and a functor to the category D. I will call these functors P subscript C and P subscript D. These are the projection functors P subscript C of C comma D is going to be C, P subscript C of f comma g is going to be f, P subscript D of C comma D is going to be D, P subscript D of f comma g is going to be g.

So, this defines, you should check that this defines functors from C cross D to C and D and moreover this category is going to satisfy the universal property for products in the category of all categories.

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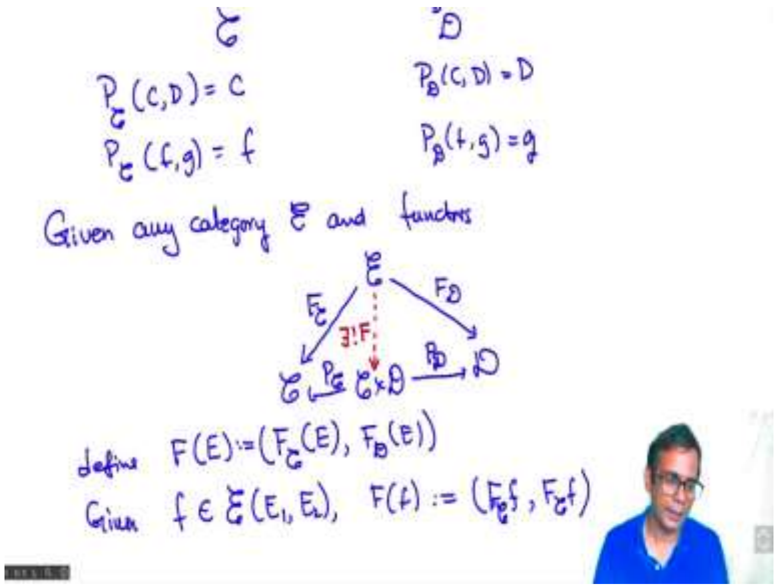

So, given, any category E and functors. So, let us just take from E, suppose we have two functors. Let us say F C and F D to C and D then we also have this C cross D over here and we have factors P D and P C. We can construct a functor here F, and how do we construct F?

Define F of any object E to be F C of E comma F D of E and given any morphism let us say we have two objects E 1 and E 2 of E we can define F of f to be, F E of f comma, F C of f comma F D of f. So, that will give us a functor from the category E to the category C cross D and this is forced for this diagram to commute and you can check. So, therefore that this category C cross D satisfies the universal property for products in the category of all categories.