Algebra-II
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Lecture 52
Functors

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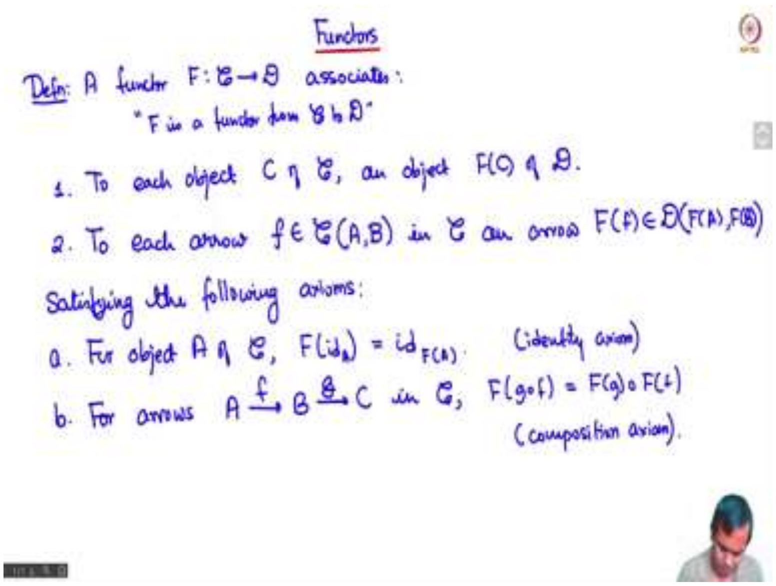

We are not quite familiar with the concept of a category. A category has objects and it has arrows and there is a composition rule for arrows. The next concept in category theory is that of a functor. So, let me give you the definition of a functor. A functor F, so now functor relates to category C and D, so what this notation means is that F is a functor from category C to category D Associates.

So, first thing it does is to each object C of C it associates an object F C of D and the second thing it does is to each arrow let us say f in C and if you have any two objects A B, it associates an arrow F of f which is an arrow in the category D and it goes from the object F of A to F of B. And these two pieces of data must satisfy the following axioms.

Basically these axioms say that they must preserve the category's structures. So, the first is that for every object A of C F of identity of A, so remember each object has an arrow from and to itself, from itself to itself which is called the identity arrow and this should be the identity arrow of F of A. And the second is the composition axiom so this is called the identity axiom and the second is the composition axiom which says that for arrows.

So, suppose you have A, B and C three objects in the category C and you have arrows f and g in C, F of g circle f should be f of g circle F of f. This is called a composition axiom. So, basically a functor from one category to another is defined on two levels, it is defines on the level of objects and on the level of arrows and on the level of arrows it needs to satisfy two properties. It should take an identity arrow to the identity arrow of the corresponding object and it should preserve composition of arrows.

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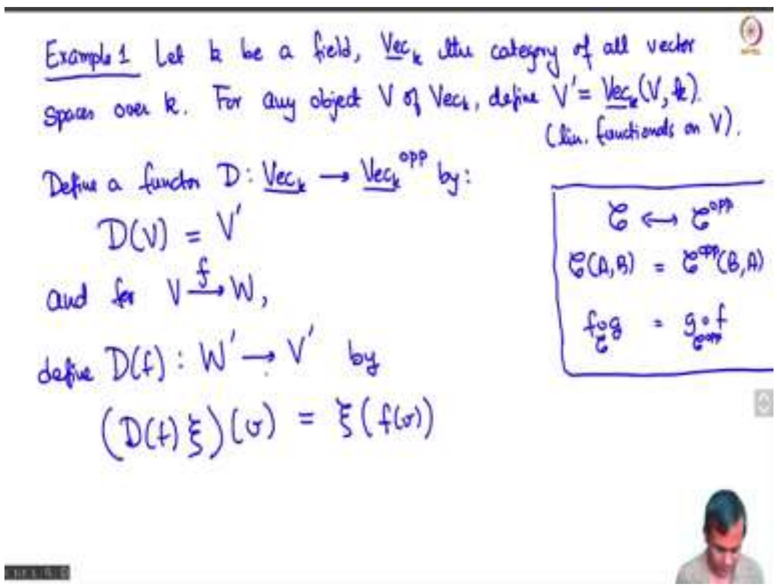

Let us look at some examples. So, let k be a field and let us write Vec k the category of all vector spaces over k. So, remember in this category the objects are vector spaces over k and given to vector spaces V and W there arrows from V to W are all the linear trans, k linear transformations from V to W.

Now for any object V of Vec k define its dual which we denote by V prime to be the set of arrows from V to k itself. So, k is a vector space of over k, it is a one dimensional vector space, and we just take all the arrows from V to k. Which in another words is just the linear maps from V to k or which we also call linear functionals on V.

Now, I will define a functor which I will denote D from Vec k to Vec k opp. So, let me remind you if you have a category C you also have a category C opp associated to it, which is called the opposite category and its objects are the same as the objects of C, but the morphisms are reversed. So, the arrows from B to A in the opposite category are by definition the arrows from A to B in the original category and composition is also reversed.

So, f circle g in C is g circle f in C opp, and this makes sense because of the way arrows are defined by reversing the direction of the arrows. So, now we are going to define a category, a functor from the category Vec k to its opposite category, such a functor is sometimes called a contra-variant functor when it goes to the opposite of another category, but in any case let us go on with the definition as follows.

So, now firstly I need to define the functor at the level of vector spaces. So, I need to say what is D of a vector space, so D of V is going to just be V dual and then I need to define what D is on the level of arrows, and so what I need to say is what is D of f so maybe let us just say for each f, from a vector space V to a vector space W suppose you have a linear, k linear map f then I need to define D f. So, that should go from W dual to V dual. Now, from V dual to W dual, because it is a functor to the opposite category.

So, this would be Vec opp of V prime comma W prime, but then it is just a linear map from W prime to V prime. So, we will define this by the following equation D f, so now it should take linear functional in W. So, let us say xi, where xi is a linear function on W and it should give me a linear functional on v. And there is only one way to put these three symbols f, xi and v together and it to say f v. So, that is a definition of D f from W prime to V prime, and now we need to check that this is a functor.

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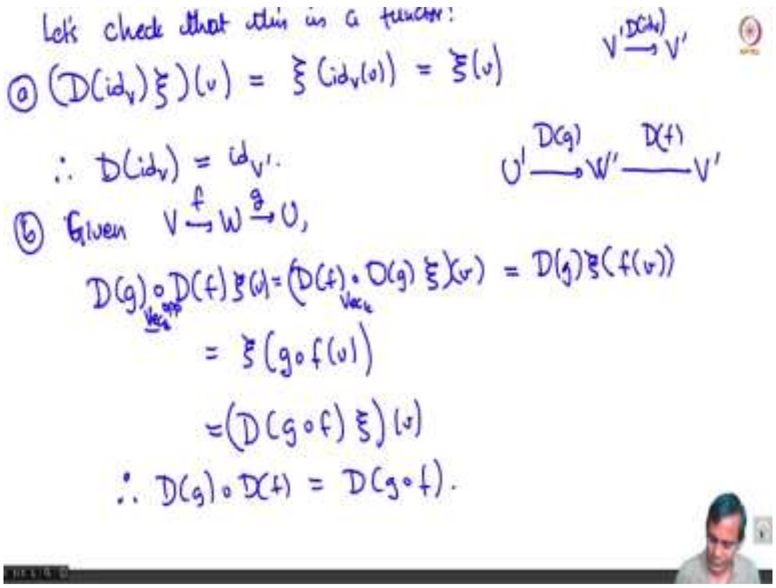

So, let us check the axioms. So, the first thing I need to check is whether D takes the identity to identity. So, let us look at D of identity of, so now we have let us say identity of V is a vector space so D of, identity of V, so identity of V is from V to V and D of identity V is from V prime to V prime. And so we need to say what it does to xi prime in D prime, maybe here I should say for all v in V, xi in W prime.

So, this evaluated at a vector v now, xi is in V prime and this is just going to be xi of identity of v which is going to be xi of v. So, that means that D of identity V is equal to identity of V prime, because it is taking xi, the linear functional xi is going to the linear functional xi. So, that checks axiom a, and let us check axiom b, which is by far the most interesting part here.

So, let us look at suppose we have arrows V W U, let us call them f and g what we want to check is whether D of g circle D of f is equal to D of g circle f. So, let us look at D of g circle D of f and let us apply it to, so D, so we have like U prime, so we have D g goes from U prime to W prime and D f goes from W prime to V prime that is the arrow reversing property of D works.

So, let us look at D g circle D f but this is the composition in the category Vec opportunity, right? And that by definition of the opposite category is D f circle D g and this is the composition in Vec k. So, let us now apply this to some linear functional xi and then let us see how it evaluates on a vector V.

So, by definition this is D g xi evaluated at f of v, but that is the same as xi evaluated at g circle f of v again by applying the definition of D but that is by definition, the same as xi, D g circle f xi evaluated at v and so we conclude that D g circle D f is D g circle f, and hence D is a functor. This D is an example of an arrow reversing, it s functor from vector spaces to the opposite category of vector spaces so it reverses arrows and it is called and contra-variant functor. So, let me just define covariant and contra-variant functors, so this is just a notational thing.

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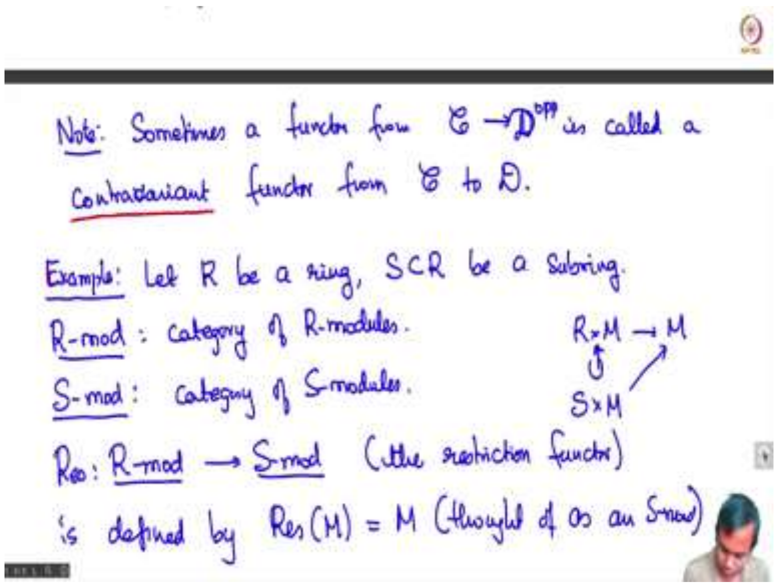

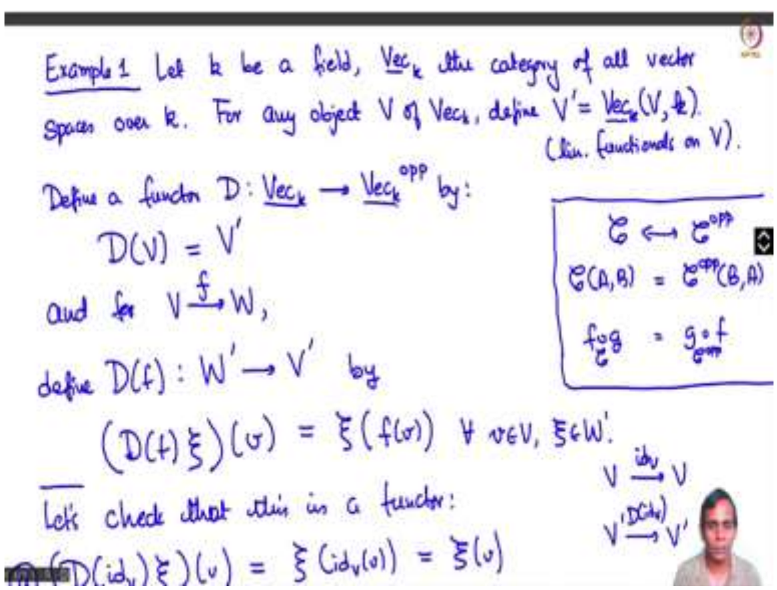

A contra-variant functor sometimes a functor from a category C to D opp is called a contra-variant functor from the category C to D. So, instead of saying there is a functor from C to D opp, we say it is a contra-variant functor from C to D so this functor, capital D that we defined earlier this is a contra-variant functor from Vec k to Vec k or you can just say it is a functor from Vec k to Vec k opp. That is the notion of contra-variant.

Now, let us look at more examples of functors. So, let R be a ring and S be a sub ring of R. Now let R-mod denote the category of R-modules, so an R-module recall is just an additive abelian group together with an operation from R cross M to M which takes r comma m to r dot m and this satisfies some nice properties which relate ring structure on R to abelian group structure on m. So, you can review this in Algebra-I so you have the category of all R-modules.

The objects are R-modules and arrows between two R-modules are R-module homomorphisms and we will say S-mod for the category of S-modules. Now the thing is an R-module can be thought of as an S module because S is a sub ring of R. So, you can just take the operation from R cross M to M which defines the R-module structure on M and then you restrict it.

So, you have S sitting inside, so S mod M is sitting inside R mod M and so this is a subset here you just restrict it and you get action of S on M, and so an R-module is automatically an S-module. And this is called the restriction of structure. So, we have a functor Res R-mod to S-mod, it is called the restriction functor, is defined by Res of M is equal to M but now thought of as an S-mod.

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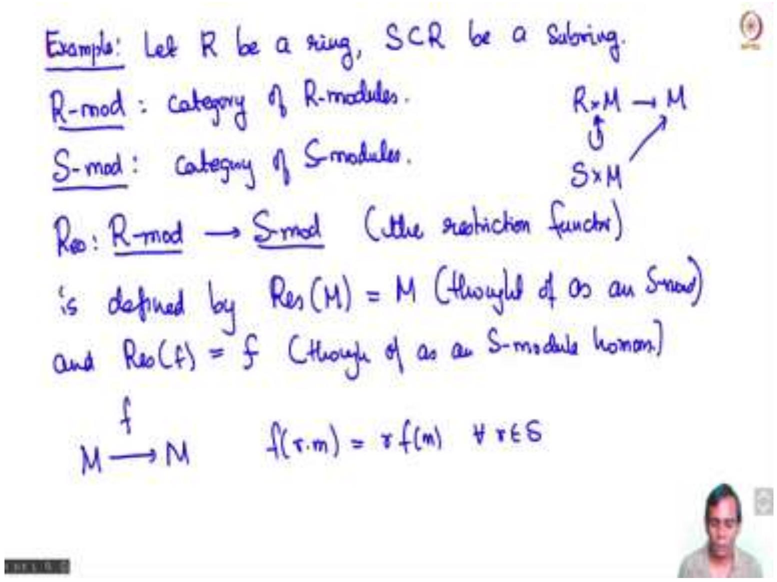

And Res of f is equal to f now if f is in R-module homomorphism from M to M that just means that f of r dot m is r dot f of m for all r in R, but since S is a subset of M, this is true for all r in R, this is also true for all r in S and so f is also an S-module homomorphisms. Every R-module homomorphism is S-module homomorphism, so you can just define, but now this is thought of as an S-module homomorphism, and this is the restriction functor from the category of R-modules to the category of S-modules.

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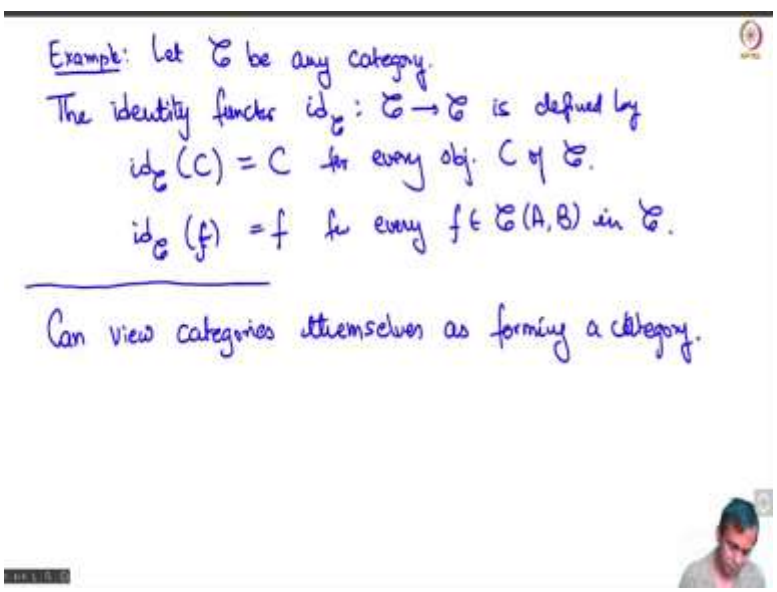

Let us look at a very simple example for functor. Let C be any category then define a functor from C to C which is called the identity functor. It is defined by setting identity C of C is equal to C for every object C of C and identity C of f is equal to f for every arrow in C. And it is very easy to check that, this actually satisfies the axioms in the definition of a functor.

This is called the identity functor and this suggests that we can view categories themselves as forming a category. The objects of this category of categories are categories, the arrows in the category of categories, are functors from one category to another and we have just constructed the identity arrow from each category to itself. However, you need to be a little careful, you could run in to set theoretic issues related to Russell's Paradox.

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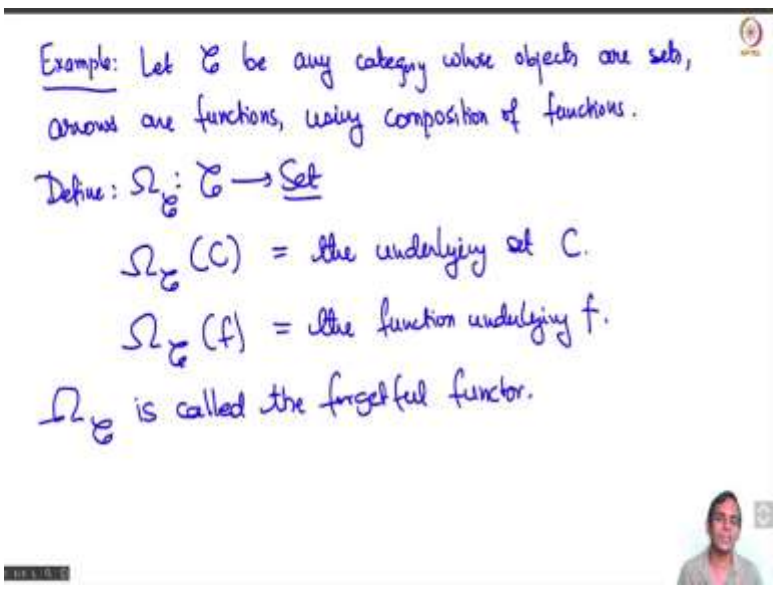

Let us look at one more example. So, now let C be any category whose objects are sets and arrows are functions and the composition of arrows is composition of functions. So, I will just say any category whose objects are sets, arrows are functions and composition is composition, composition of functions.

We have seen lots of example of this and most categories that we study in Algebra or topology are of this form. So, for example, we have the category of groups, we have the category of sets, we have the category of rings, we have the category of topological spaces where arrows are continuous functions and so on.

Then define a functor which, from C, maybe I will give it a name, I will call it omega of C, it is from C to the category of sets, which takes, for every object C the underlying set C. So, C a priori is an object to the category C but I just stipulated that every object to the category C should be a set and omega C of f is the underlying function, the function underlying f.

More explicitly if f is an arrow from C to D it is an arrow in the category C then this f is actually a function from the set C to the set D and this functor omega is called the forgetful functor. Now, the idea is that the category C, its objects are set, but they are sets with some additional structure so for example if C were the category of groups then the category C is the category of sets together with the additional structure that is the binary operation on the group.

And this must of course satisfy some axioms and so on. And what this functor omega C does is it forgets all that additional structure and only remembers the fact that you have a set. This functor has many beautiful applications we may see some in a later lecture. But for now it is just a nice example of a functor. I will let you check that this is actually a functor.

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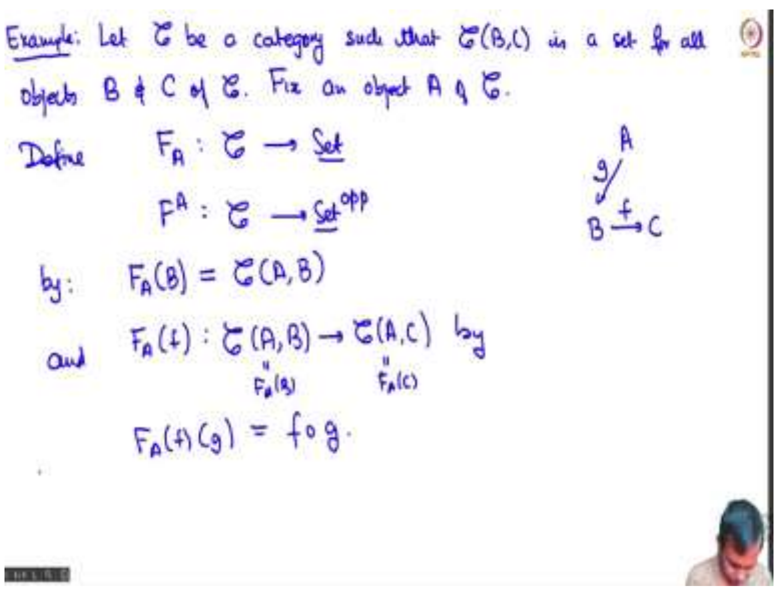

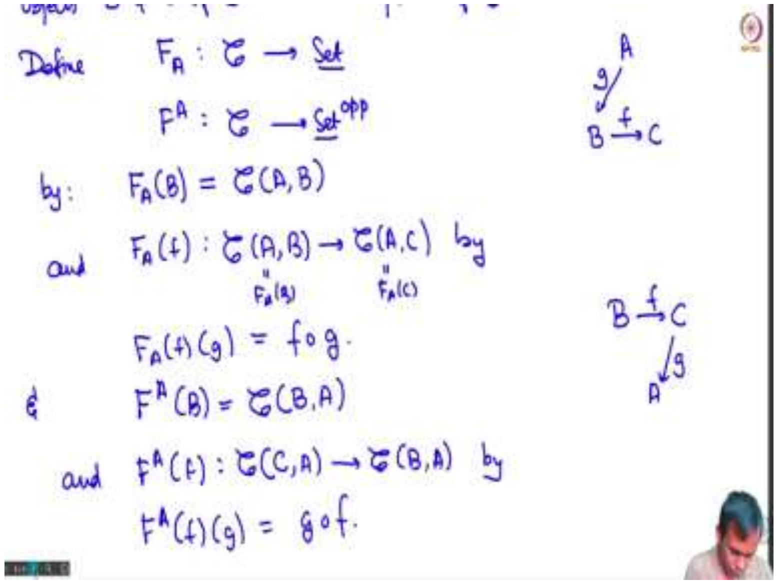

The next example, consider a category C with a mild technical assumption we require that the collection of arrows between any two objects is a set. This is not a very restrictive condition, all the categories of the form that we studied in the previous example namely categories where objects are sets and arrows are functions would satisfy this axiom and unless you really worry about the intricacies of set theory I will say not to pay much attention to this as of now.

Now, fix an object A of C. Now with respect to this object fixed object A I will define two functors from C to Set. So, the first functor I will denote by F subscript A it is from C to Set and the second functor is F superscript A it is also from C to Set but it is a contra-variant functor so I will call it as a functor from C to the opposite category of Set.

As follows, so firstly I will define F subscript A, so F subscript A of B is defined to be all the arrows from A to B and I need to define it on the level of arrows so suppose now I have B to C I have f then, so this should be a function from C A B to C A C, this is F A of B, this is F A of C. So, I need to define a function from F A of B to F A of C.

So, if I have arrow g from A to B I need to somehow construct an arrow from A to C and there is only one way to do that which is to take f circle g. And similarly we will define F superscript A. It is sort of the dual notion. So, we are just going to reverse the arrows and F superscript A of f so now this is going to be C from, so now F is B to C as before and then we have from C, now because it is going to the opposite category this will be from C A to C B A, which means that now we are given something from C to A.

We need to construct something from B to A and there is only one way to do that, which is to take g circle f. This is defined by F A f of g is g circle f. Now we need to check that these are functors, of course I have defined them on the level objects and I have defined them on the level of arrows, just as an exercise let us check that one of them satisfies the axiom, so the functor I will leave the other for you to do yourself.

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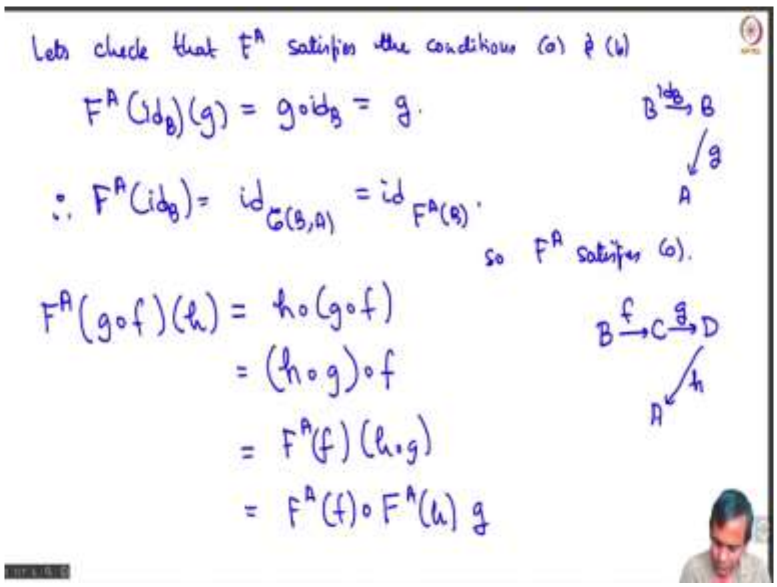

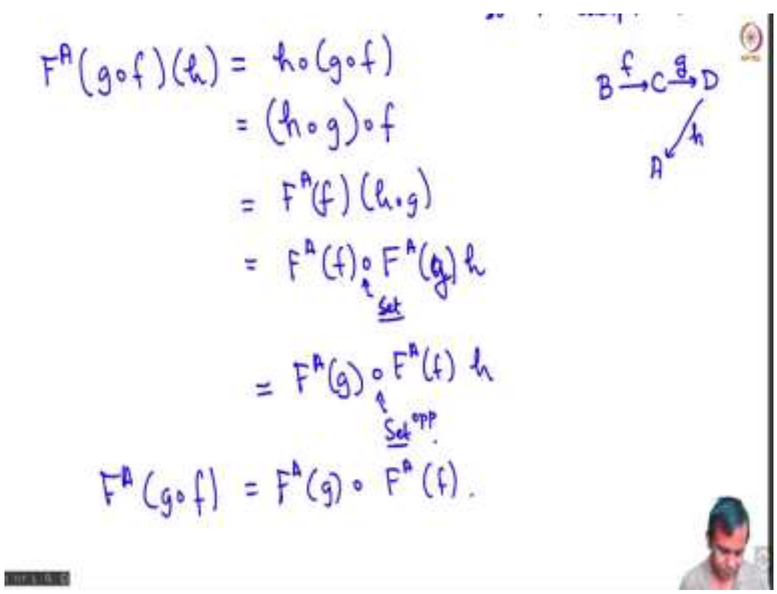

So, let us check that F superscript A satisfies the conditions a and b in the definition of a functor. So, the condition a was that the identity arrow goes to the identity arrow so F superscript A of the identity of B we have some element B so now we have a situation and we have g here and so this is by definition g circle identity of B and so this is g, which means that F circle A of the identity of B is the identity function of the set B to A, which is the identity function of F superscript A of B, and that is the identity axiom.

So, F A satisfies the axiom a, and let us check that F A satisfies the axiom b. So, now the things get a little more complicated. We have B to C to D we have three objects and we have two morphisms, let us call them f and g and now we have h from D to A. So, let us look at what is F A of g circle f evaluated at h. Well this is by definition h circle g circle f, but by associativity of composition I can write this as h circle g circle f. That is F A f applied to f circle g and so that is F A f applied to F A h applied to g.

But this is composition in sets in the category set and, so in the category Set opp this is a composition of function. So, in the category Set opp this is F A, oops! This is g applied to h, and this is composition in the category Set opp. And so what we have is F A g circle f is F A g circle F A f. So, F superscript A is indeed a functor. I will leave it as an exercise and I would highly recommend it that you sit down and write down a careful proof that F lower A is also a functor.

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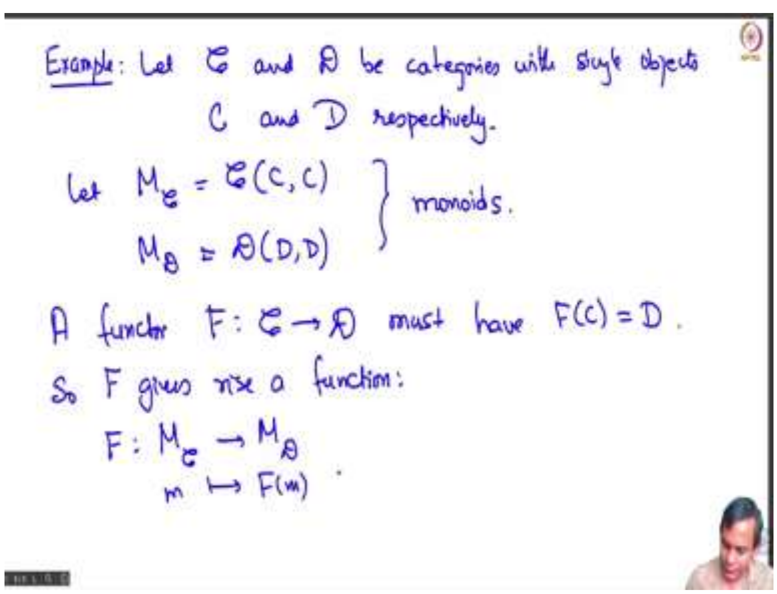

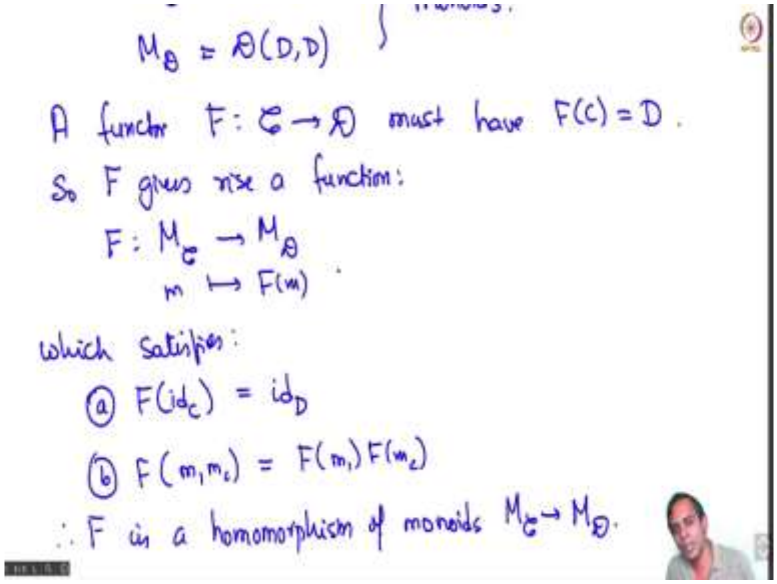

Now, one last example of a functor. We will look at categories with one object. So, let C and D be categories with single objects. Let us say C and D respectively. The category C has a single object which I denote by C and the category, script D has single object which I denote by ordinary D, and let M C denote the set of arrows from C to C and let M D denote the set of arrows from D to D.

And what you have seen is that these are actually just monoids, because the definition of a category says that the set comes with a composition operation and an identity so these are monoids and a functor F from C to D are must satisfy, there is only one object so it must take this object to the only object of D, and so the only question is what does it do on arrows.

So, what we get is F gives rise to a function, F which I will also denote F from M C to M D which is just f goes to, okay maybe I will call it small m close to F of m. And this function must satisfy the axioms for a category which is that F of identity of C, which is the identity of the monoid M C is the identity of D which is the identity of the monoid M D and the second is that F of m 1 m 2 is equal to F of m 1 F of m 2.

So, what we are seeing is that F is a monoid homomorphism. So, functors between single object categories are homomorphisms of the corresponding monoid. If these categories, if these monoids were actually groups then they would be group homomorphisms.