Algebra-II
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Lecture 51
Categories: Second Problem Session

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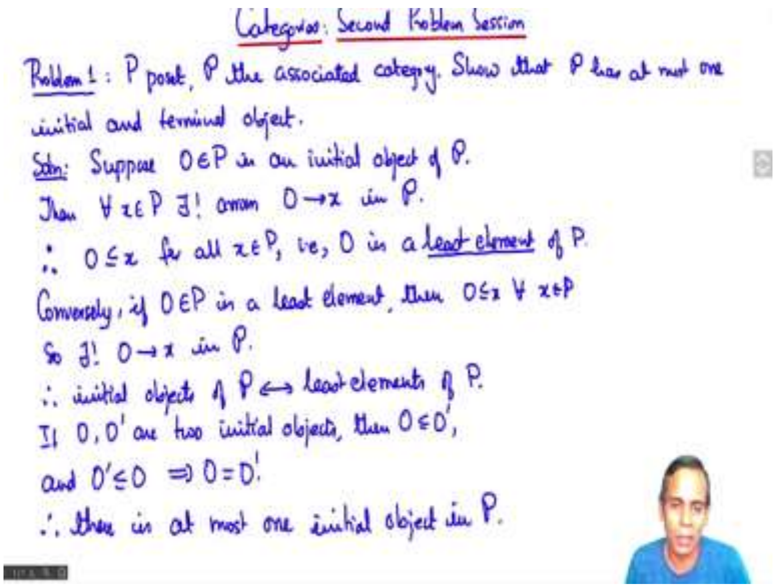

Let us solve some problems. Suppose we have a poset P and script P is the associated category. Then show that the category script P has at most one initial and terminal object. At least at most one initial object and at most one terminal object. So, you can pause your video here as you try to solve it. Otherwise, you can watch me solve it.

So, in order to solve this problem we need to understand what is an initial object in this category script P. Suppose 0 is an initial object. So, 0 is an object of script P so it is an element of P, is an initial object of P. Then what we know is by definition of an initial object there exists a unique arrow, maybe I should say for every x in P.

There exists a unique arrow from 0 to x in P, okay but by definition of P this means that 0 is less than or equal to x for all x in P, therefore the 0 is an element of the poset which is less than or equal to all the other elements of P, in other words 0 is what is called a least element of P. That is just the definition of least element, and conversely if 0 is a least element, then of course, in this category between any two objects there is at most one arrow.

So, then there is a unique arrow, well if, so then 0 is less than or equal to x for all x in P and so there exists a unique arrow 0 to x in P. Therefore initial objects in the category script P are precisely the least elements of the poset P. We need to show that there is at most one least element in P. So, if 0 is and 0 prime are two initial objects, then 0 is less than or equal to 0 prime because 0 is the least element of P and 0 prime is less than or equal to 0 because 0 prime is the least element of P.

But by the axioms of a partially ordered set this implies that 0 is equal to 0 prime. Therefore, there is at most one least element in P and hence there is at most one minimal element in P, initial object in P. A very similar argument can be used to show that there is at most one terminal object in P.

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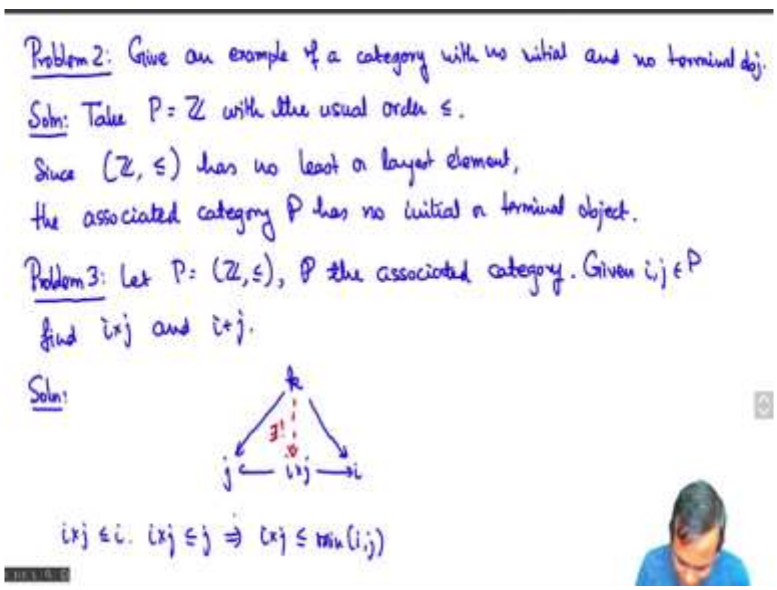

Problem two, give an example of a category with no initial and no terminal object. Try to find an example yourself, otherwise you can watch for the spoiler ahead. So, there are many such examples but we will give an example in the spirit of problem one. We have seen in problem one that an initial object in a category associated to a partially ordered set is a least element of that partially ordered set.

And if you followed up problem one for terminal objects then you would have realized that a terminal object in the category associated to a partially ordered set is a largest element in that partially ordered set. So, what we could do is we could just look for a partially ordered set which has no least elements and no largest element and a very simple example of that is the partially ordered set consisting of integers with the usual order.

So, since this partially ordered set has no least or largest element the associated category P has no initial or terminal object. And you can modify this example to get a category where there is an initial object and no terminal object. For example take the non-negative integers or you could construct an example where there is no initial object but there is a terminal object, taking for example the non-positive integers.

Now, let us move on to problem three, in the category P associated to Z. So, let P be Z with the usual order and P the associated category, then find, so given integers i, j in P find i cross j and i plus j namely the product and the co-product in P. To solve this problem let us see what we require of a product of i and j. So, we have a product of i and j then it should have arrows to i and arrows to j.

So, in particular i cross j needs to be less than or equal to i and i cross j needs to be less than or equal to j, which means that i cross j needs to less than or equal to minimum of i and j. And so that is the first requirement and then the second requirement is that if you have any object k in your category with arrows to i and j, then there exists a unique arrow from k to i.

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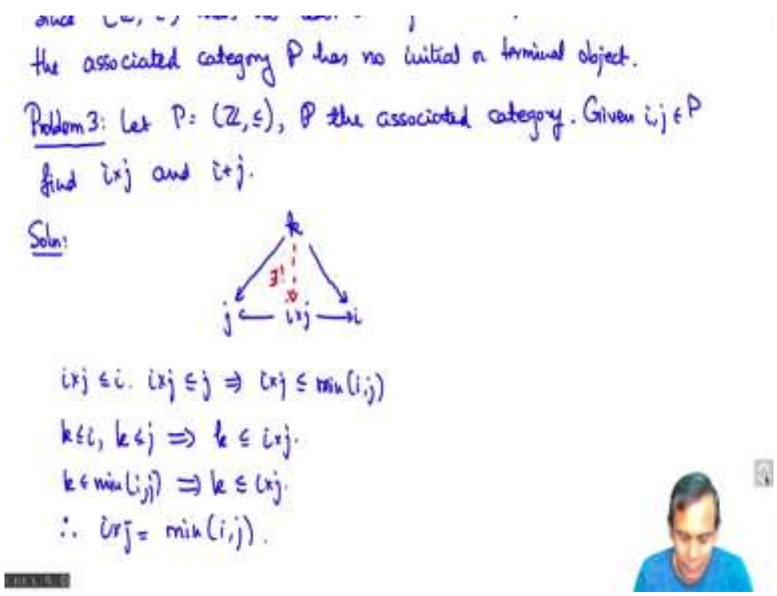

Now if k has an arrow to i and j so the second condition is that if k has an arrow to i then that means k less than or equal to i and the second condition means that k less than or equal to j implies that k is less than or equal to i cross j. So, what we are saying is that, so this condition here is saying that k less than or equal to min i j implies that k is less than or equal to i cross j.

So, the first condition implies that i cross j is less than or equal to min i j and the second condition implies that i cross j is greater than or equal to min i j. So, all in all what we are saying is that i cross j is equal to min i j. A very similar reasoning with all the arrows reversed will show you that i plus j is the max of i and j.

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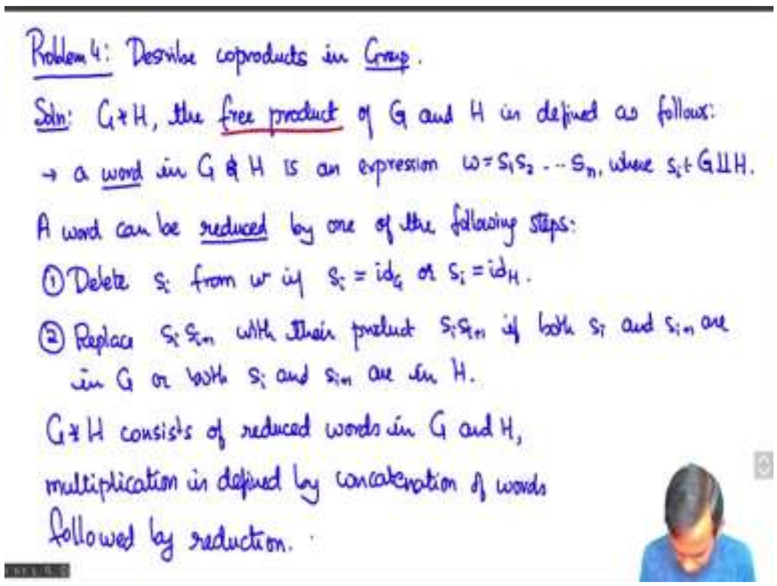

Problem four, describe co-products in the category of groups. This problem is a slightly technical problem for those who are familiar with groups. So, it is also a little hard, let me explain the solution but it would be a good idea to think about it a little bit, at least to appreciate where the difficulties of this problem lies.

So, recall that the category of the group, of groups is the category whose objects are groups and whose arrows are group homomorphisms. So, now the co-product in the category of groups is what is called the free product of groups. So, G star, it is usually denoted G star H, the free product of G and H is defined as follows.

So, it is, the definition itself is a little complicated. So, firstly a word in G and H is an expression of the form w equals s 1 s 2 s n where each s i belongs to G or belongs to H. Okay so each s i is either an element of G or an element of H. So, that is a word and a word can be reduced by one of the following steps.

So, these are rules what you are allowed to do to a word to reduce it. So, the first one is replace or rather delete s i from w if s i is equal to identity of G or s i is equal to identity of H. That is the first thing you are allowed to do to reduce a word and the second is replace two letters in the word s i s i plus 1 with their product. s i s i plus 1 if both s i and s i plus 1 are in G or both s i and s i plus 1 are in H.

So, if you have two adjacent letters in your word which are from the same group you can take those two letters, multiply them together, so erase, remove those two letters and replace them by their product and so each time you reduce a word its length goes down and therefore this product of reducing a word must stop after finitely many steps, leaving you with a word which can no longer be reduced.

This might be a word with no letters at all namely the empty word or it could be a word which cannot further be, a non-empty word which cannot be reduced. So, the group G star H the elements of the group G star H consists of reduced words in G and H. What does the reduced work look like? Well a reduced word looks like s 1. So, it would start with say some element of one of the groups.

So, say it starts with an element of g, so it will be g 1 but then this g 1 cannot be the identity of G. The next letter has to be from the group H so it has to be h 1 g 2 h 2 and then it can either end with g or with h. So, an example is something like this g k h k so this is an example where g 1 g 2 g k are non-identity elements of G, h 1 h 2 h k are non-identity elements of H.

So, a reduced word is a word whose letters are non-identity elements of G and H alternating with each other. Okay so G star H consists of reduced words in G and H and multiplication is defined by the group operation, is defined by concatenation of words. So, you just write one word and write the other word next to it followed by reduction.

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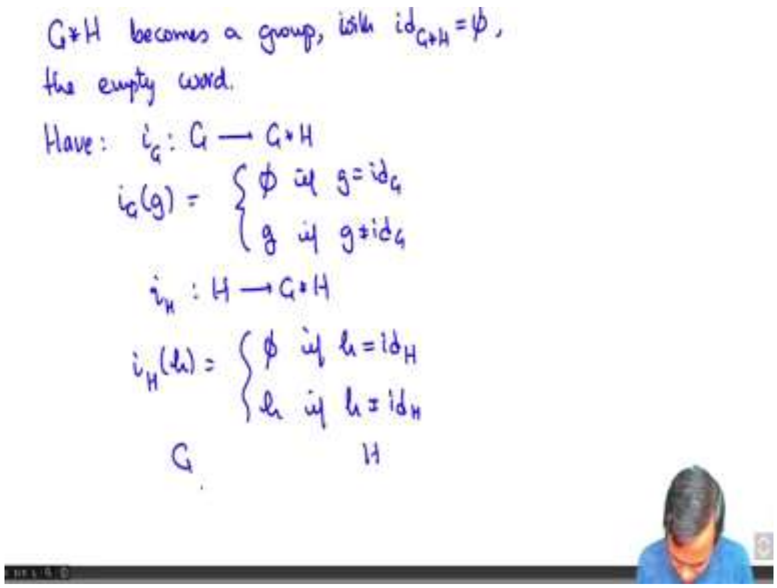

So, then G star H becomes a group. The identity element of this group is the empty word and now you have homomorphisms from G to G star H defined by i G of g is equal to empty word if g is the identity and g, the single, the word of length one, with just one letter, if g is not identity and similarly identity of H, I, the map from h is empty word if h is equal to the identity of H and if h is not equal to the identity of H. So, this defines map i H from H to G star H.

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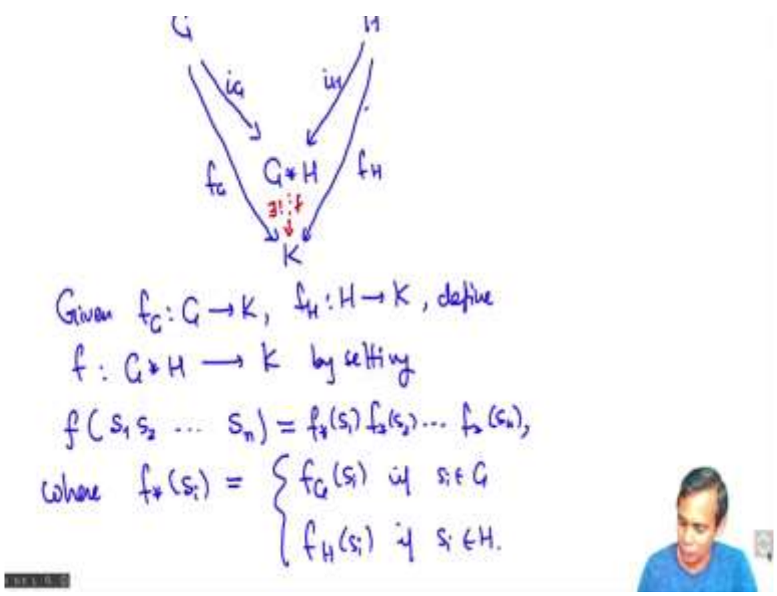

And so now we have this, we have G, H and then we have G star H, I G and i H. What we need is that suppose we have some other group K and we have a group of morphisms from, let us just write this up here, and suppose we have group homomorphisms f G and f H from H to K, G to K and H to K then we want that there exists a unique group homomorphism from here to here, f.

So, given f G G to K, f H H to K, define f from G star H to K by setting. So, I need to tell you what f does to any reduced word in G star H. So, f will be applied to the word s 1 s 2 s n and should give me an element of K and I define that as follows I define it to be f star s 1, f star s 2, f star s n. I need to tell you what f star is, where f star s i is equal to f G of s i if s i belongs to G and is equal to f H of s i, if s i belongs to H.

And you can show that this f will be the unique group homomorphism from G star to K which makes this diagram here commute and thus a co-product in the category of groups is what is called the free product of groups. For those of you who were in Algebra-I you have already seen an example of a co-product before.

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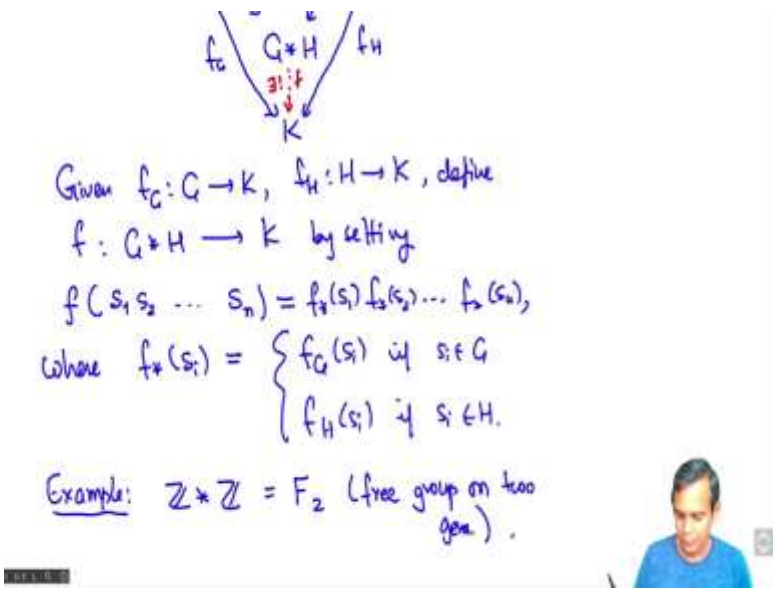

If you take the group Z, the cyclic group Z, the infinite cyclic group, its co-product with itself is the free group on two generators.