Algebra-II
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Lecture 50
Products and Coproducts

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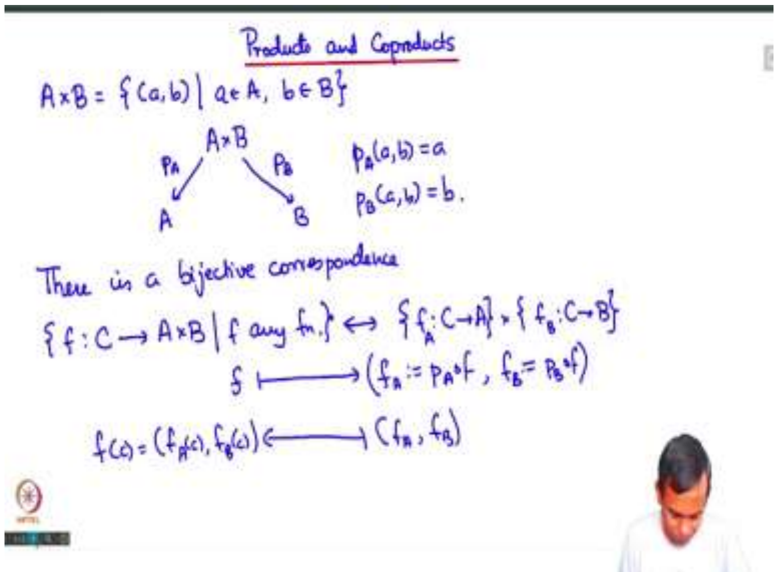

In this lecture we will learn about the definitions of products and co-products which are the first sort of definitions by universal construction in category theory. Now, the idea behind products is the Cartesian product of sets. So, recall that if you have two sets A and B then a cross B is just the set of all pairs, ordered pairs a comma b where a is in A and b is in B.

And this set comes with two projection functions, so I will just visually draw it like this A cross B to A, there is a projection function called P subscript A and from A cross B to B is a projection function called P subscript B, and these are defined by P subscript A of a comma b is a, P subscript B of a comma b is equal to b.

Now, these projection functions have a nice, give rise to a nice correspondence, so there is a bijective correspondence between the set of all functions from C to A cross B, and the set, well the product of sets f is a function from C to A, maybe I will call it f subscript A cross f subscript B C to B. Namely given any function f from C to A cross B, I can map it to f sub A, which is defined to be P subscript A circle f comma f sub B, which is defined to be P subscript B circle f.

And conversely given any pair f sub A comma f sub B, I can associate to it a function which takes c to f subscript A c, f subscript B c. So, f subscript A c is an element of A, f subscript B c is an element of B, so this is an element of A cross B and this is a bijective correspondence. Somehow in category theory this kind of a correspondence captures the essence of products.

So, the game is somehow to take the notion of a product and rewrite it in such a way that we do not talk about elements of sets, where we only talk about objects and arrows. We would like to rewrite the notion of a product purely in the language of category theory.

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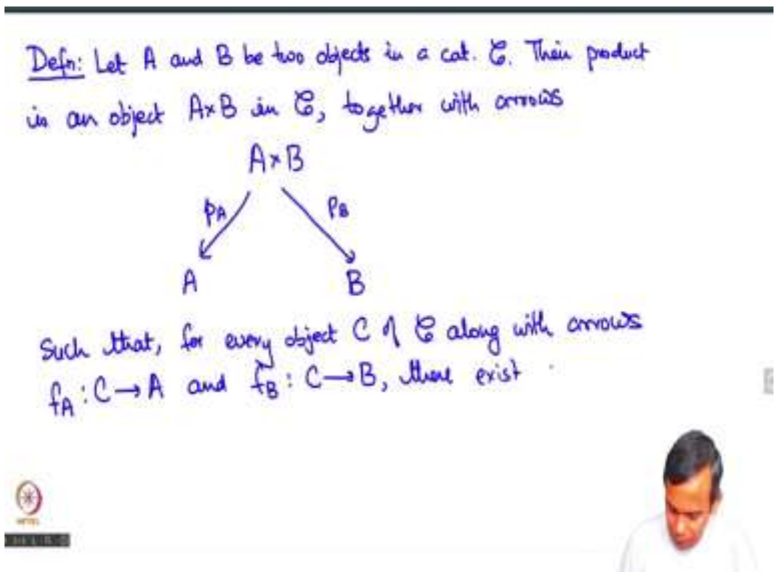

So, let us do that and that definition is the category theoretic definition of a product. So, let A and B be two objects in a category C then their product A cross B is an object, well maybe I should say their product is an object. A cross B in C which comes together with two projection maps so I will, just two arrows together with arrows, and I will use diagrammatic notation.

So, I will just say A cross B, we have an arrow P subscript A and P subscript B, this goes to A and this goes to B, such that for every object C of the category C along with arrows f subscript A from C to A and f subscript B from C to B, there exists. Now, let me explain this whole thing diagrammatically.

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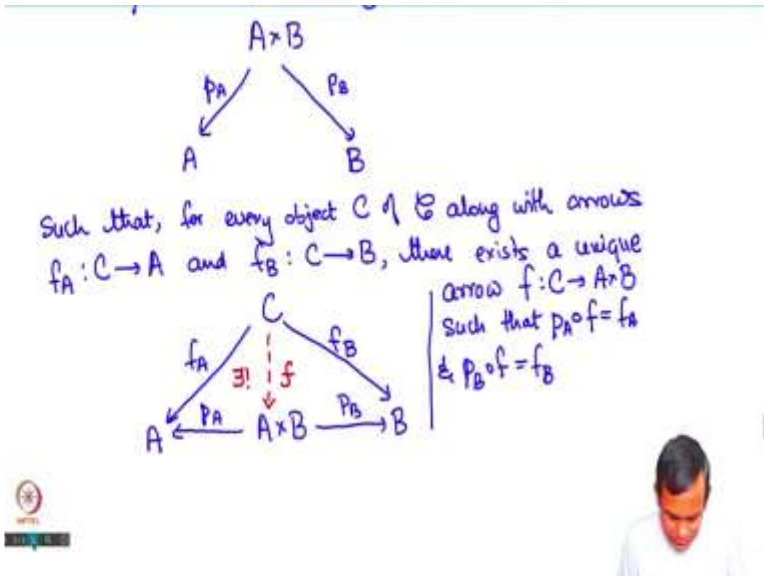

There exists, so I will draw this line so we have A cross B, we have P subscript A, the projection on to A, we have P subscript B, projection on to B. I am using the words projection and so on but remember A and B are no longer sets, they are just objects in some category. And A cross B is again an object and these are just some arrows in the category.

And we have this object C and we are given functions from f subscript A from C to A and f subscript B from C to B, and what we want to say is that there exists a unique function f from C to A cross B. So, let me write it out explicitly there exists a unique arrow. I will just write it here arrow f from C to A cross B such that P subscript A circle f is equal to f A and P subscript B circle f is equal to f B.

Now, these conditions are more easily visualized in this diagram here on the right. What we are saying is in this diagram you go from C to A, there are two possible paths. One path is to go directly from C to A using the arrow f A and another possible path is to go from C to A via A cross B and so the arrow involved here is P subscript A circle f and what we are saying is that both these arrows are equal.

So, P circle A, P subscript A circle f is equal to f subscript A as arrows, this is an equality in arrows from C to A and similarly P subscript B circle f is equal to f subscript B. So, what we are using here, we are relying very heavily on a diagrammatic notation. Let me just explain this a little bit, so a small digression before we go back to talking about products.

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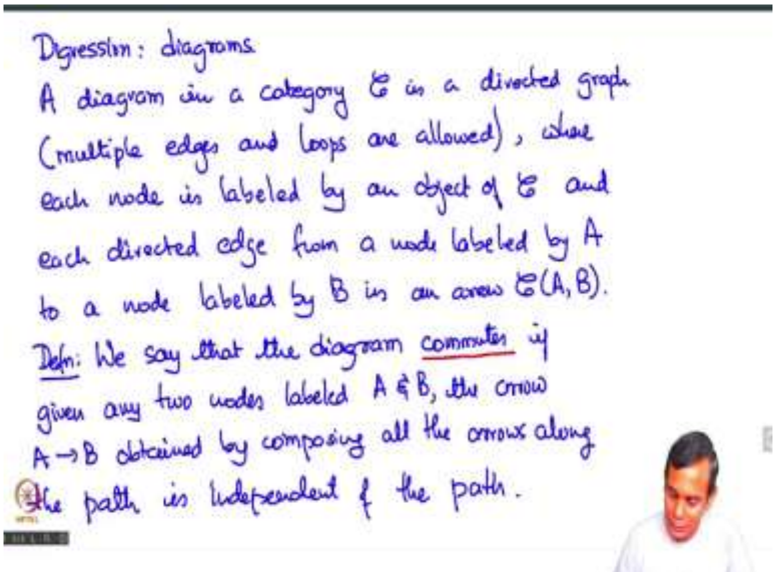

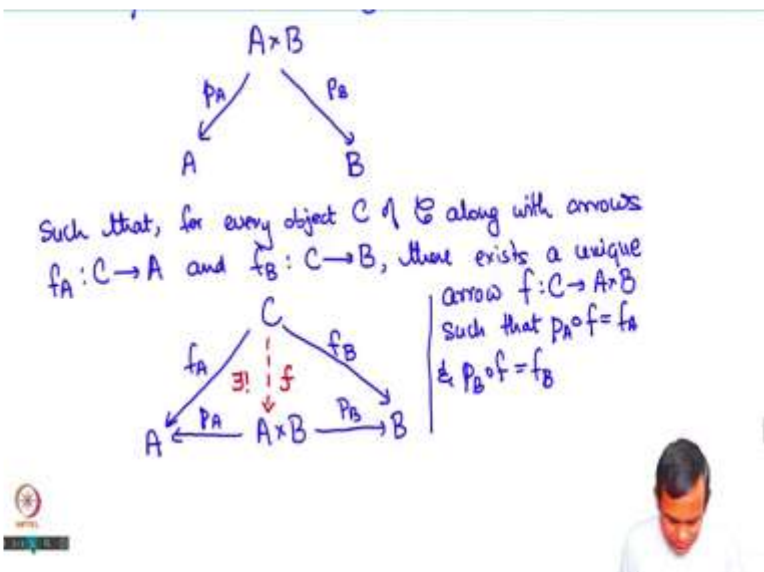

Diagrams, so a diagram in a category C is a directed graph and in this directed graph multiple edges between nodes are allowed and multiple loops are also allowed where each node of the directed graph is labeled by an object of C, and each directed edge from a node labeled by, so it will be labeled by some object say by A to a node labeled by B is an arrow in the category C from A to B.

So, this picture here is certainly such a diagram. It is a diagram. It has four nodes. They are labeled C, A, B and A cross B, and it has five arrows f subscript A, f, f subscript B, P A and P B, and we say that the diagram commutes if, well so given any two nodes in this diagram there are many different ways to go from one node to the other.

So, you can take a path and you can compose all the arrows along that path, and the resulting arrow from C to A is independent of which path you chose. So, commutes if given any two nodes, the, say nodes, say labeled A and B the arrow A to B obtained by composing all the arrows along the path is independent of the path. So in this picture for example we have that, if you take all the paths from C to B, well there are only two paths.

We have to take directed paths. One going directly from C to B and the other going from C to B via A cross B. This path from C to B, directly from C to B just gives rights to the arrow f B and this path gives rights to the arrow PB circle f. Okay so this is probably easier to understand on an intuitive level rather than writing long definitions. Nevertheless let me just explain why we use the nomenclature commutes.

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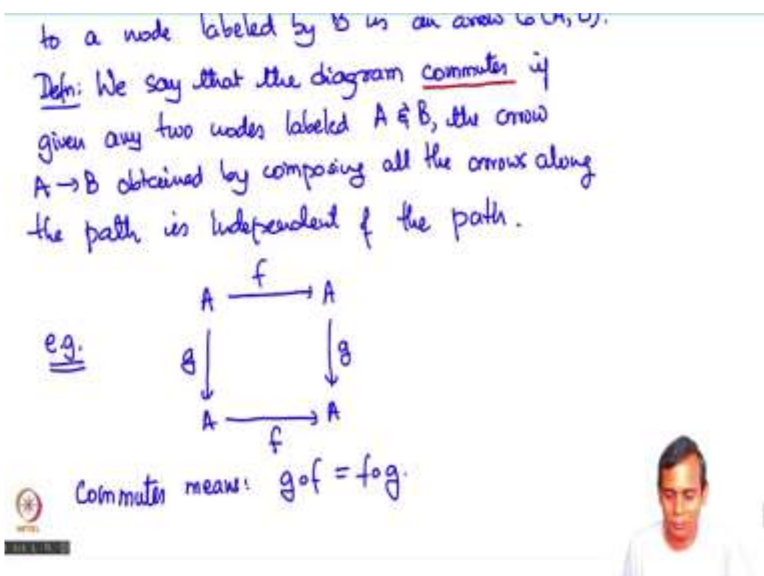

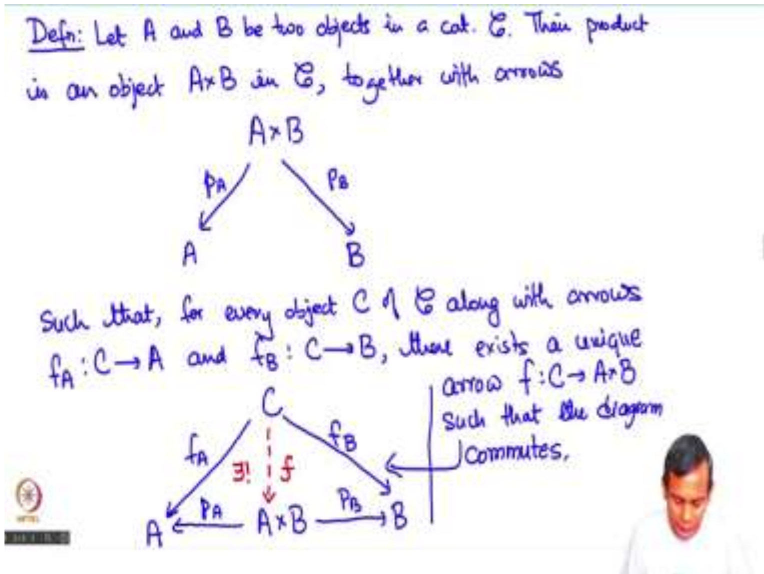

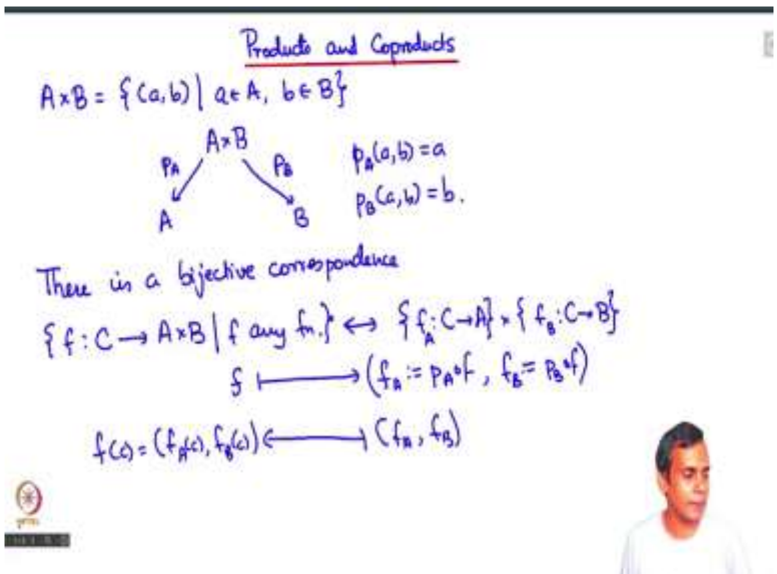

So, it comes from a very simple diagram. So, suppose I have objects A, B, B and C, and I have arrows f and g. f is an arrow from A to B and g is an arrow from B to C and well this is not going to work so well. So, what I need is, let us take all these nodes to be the same. So, I have A to A, A to A, this is probably the right way to do it.

And I have g and I have f, so if these are all arrows then this diagram commuting means that g circle f is equal to f circle g. So, the arrows f and g which are both arrows from A to A they commute and this gives you some intuition for why this is called commuting, but of course it is a much more general phenomenon.

So, coming back to the definition of a product I can try to rewrite it as there exists unique f from C to A cross B such that this diagram commutes. So that we can take to be the definition of the product of sets and in the category of sets the product is just the Cartesian product as explained in this example which we did before even talking about the category theoretic definition of products.

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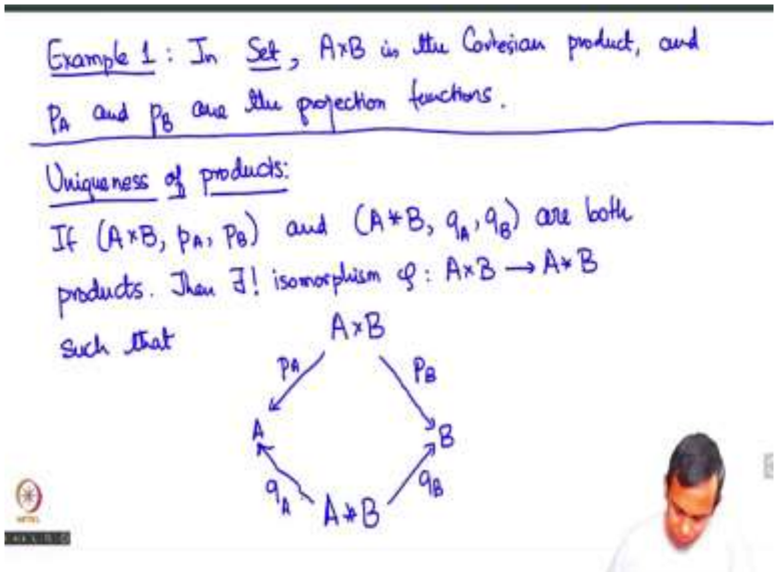

So, example 1 maybe the motivating example is that in the category Set of all sets A cross B is the Cartesian product, of two sets and P A and P B are the projection functions. Now this definition of product is a bit like the definition of an initial object. We will see this in a moment. So, the first result that we will prove about the product is that it is essentially unique.

So, here in the definition of product we have said it is an object A cross B which does not rule out the fact that there may be many ways of constructing the product, but there is a sense in which this product is unique. So, let me formulate the statement carefully and then prove it. So, this is the uniqueness of products.

So, the statement is the following if we have two different products so A cross B, remember product comes with these projections maps P A comma P B and let us say we have one other notion of product A star B so this is a different object in the category possibly and we have maybe projection maps which I will call q A comma q B, are both products in a category C of the objects A and B.

Then there exists a unique isomorphism phi from A cross B to A star B such that, and now this is best explained in terms of a diagram rather than writing down equations. So, we have this one product A cross B and we have projection maps P A to A, P B down to B, and we have another notion of product which we call A star B and we have projection maps q A and projection map q B.

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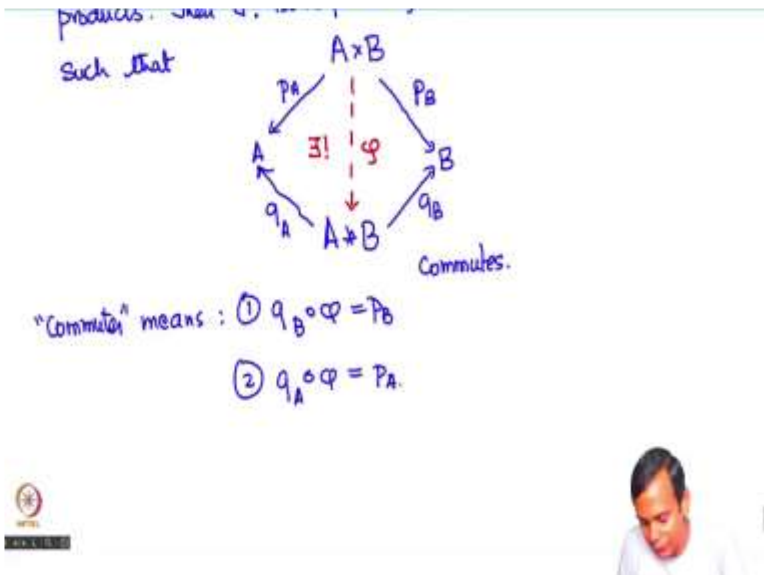

So, then the assertion is that there exist a unique isomorphism phi, such that this diagram commutes in other words you can go from A cross B to B either by P B directly or you can go via A star B. So, what this commute is saying, let me just spell it out again since these are new concepts so commutes means one that q B circle phi is equal to P B and secondly q A circle phi is equal to P A. So, these are the two assertions being made here. So that is the statement of the uniqueness of products. And now let us go on to prove this.

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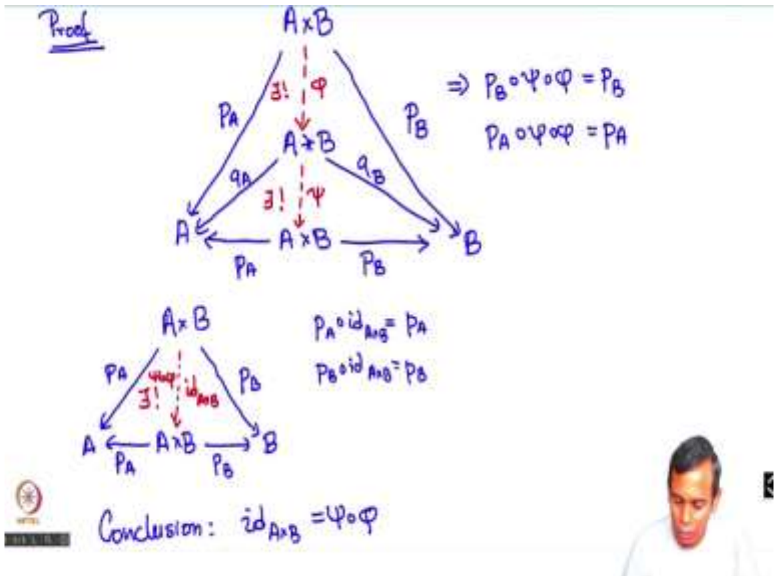

So, the prove is more of a performance than something we write down here. So, let us see, so we have A cross B and it is, we have these projection maps P subscript A to A and we have a projection map to B called P subscript B and then we have A start B and we have a projection map q subscript A and we have projection map q subscript B.

And what the universal property or the definition of the product says, the fact that A star B is a product then we can take C to be this object A cross B and we can take f A to be P A and f B to be P B in the definition of the product. Since A star B, q A, q B is a product there exist a unique arrow phi, which goes from A cross B to A star B and makes this diagram commute.

Now, let us go one more step. Let us go back to A cross B here and A cross B has arrows to A and to B namely p A and p B. Now, because A cross B is a product we can take C to be A star B in the definition of product and f subscript A to be q A and f subscript B to be q B. So, then the universal property of the product says that there exists a unique arrow psi from A star B to A cross B which makes this diagram commute.

So, the universal properties of, so this whole diagram commutative diagram comes about, because of A cross B and A star B both being products. Now let us consider a slightly different diagram a simplified version of this, so we have A cross B, we have P A to A and we have to B we have P B and we also P A and P B now the universal property for the product applied to the product itself.

So, now we are doing this, applying the universal property of the product by taking C to B A cross B itself, and taking f A to be PA and f B to be P B. so the universal property implies that there exists a unique arrow from here to here, right, which makes this diagram commute, but if we take here identity of A cross B then surely this diagram commutes right?

So, what we have is P A circle identity of A cross B, well that is just P A and P B circle identity of A cross B is P B. that is how identities work, but on the other hand from this diagram we also see that this diagram commutes if we put here instead of identity of A B we put psi circle phi. So, this diagram here commutes whether you put psi circle phi or whether you put the identity of A cross B. Now, by the uniqueness this implies that psi circle phi is identity of A cross B.

So, from this diagram, this diagram implies that P B circle psi circle phi is equal to P B and P A circle psi circle phi is equal to P A. Comparing these two we get identity of A cross B is equal to psi circle phi.

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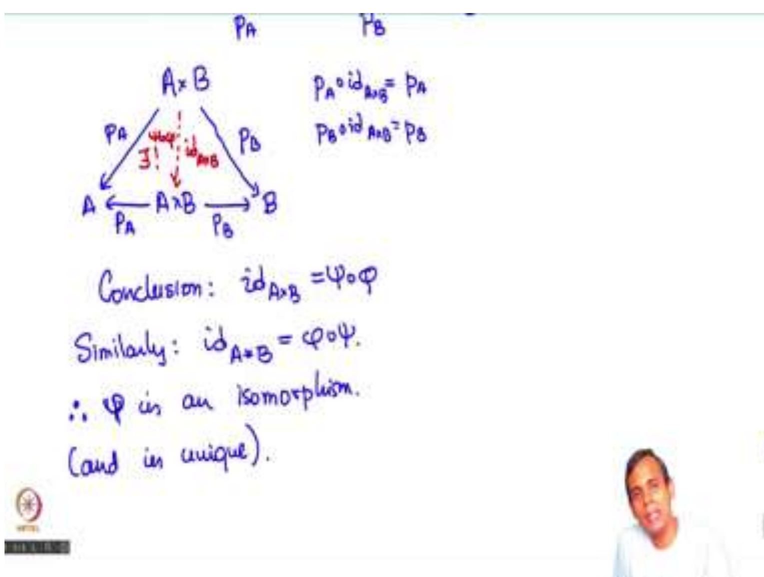

And in a similar manner where we would interchange the roles of A cross B and A star B, in this diagram we would be able to show that phi circle psi is also the identity of A star B and therefore phi is an isomorphism. It has an inverse namely psi, and of course it is unique because of the universal property we already see that it is unique here, because it is defined by the universal property. So, it has to be a unique subject to this part of the diagram commuting, that uniqueness is just coming from the universal property.

So, that concludes the proof of the uniqueness of the product. So, it is not just saying that the object A cross B is unique. It is saying that the triple A cross B together with the arrows P subscript A, P subscript B is, well it is not unique there could be different objects with this property, but those objects would be isomorphic via a unique isomorphism.

And this is just like, it is a slightly more complicated situation but it is like the situation we saw last time where we said that an initial object in a category is any two initial objects in a category are isomorphic via a unique isomorphism.

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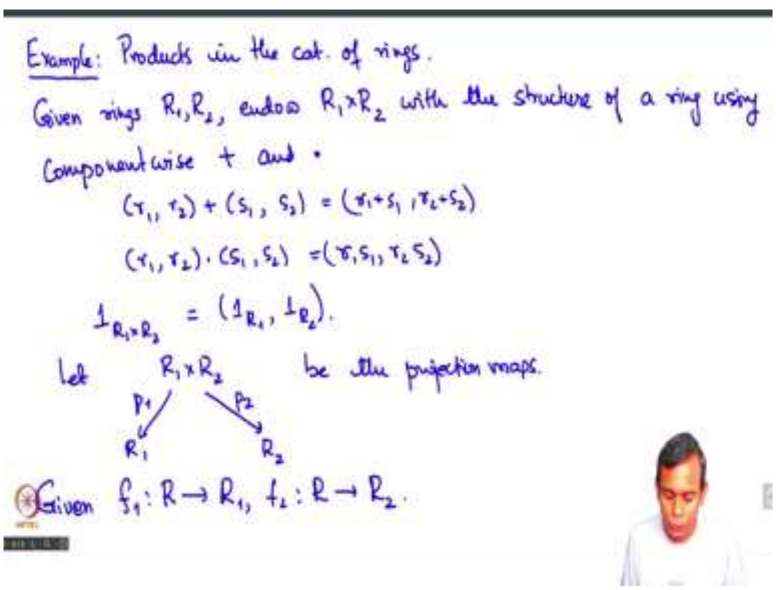

Let us look at another example of a category where products exist. Let us look at products in the category of rings. So, recall that ring is an algebraic structure with two binary operations addition and multiplication, under addition if forms a billion group under multiplication it forms a monoid and these two algebraic structures are related via distributivity laws.

We have always insisted that rings be unitive so there is a multiplicative unit in everything and the arrows in the category of rings are ring homomorphism so they preserve addition and multiplication but also they must map the unit of the domain ring to the unit of the image ring.

So, that is the category of rings and in the category of rings there is well defined notion of product so given two rings R 1 and R 2 you can take the Cartesian product R 1 cross R 2 and endow it with the structure of a ring using component wise addition and multiplication.

So, you have R 1 comma R 2 plus S 1 comma S 2 equals R 1 plus S 1 comma R 2 plus S 2 and you have R 1 R 2 dot S 1 S 2 is R 1 S1 comma R 2 S 2, and this gives rights to a ring structure on R 2 cross R 2. Note that the unit of R 1 cross R 2 is the unit of R 1 comma the unit of R 2. So, you can check that all the axioms for a ring are satisfied by this product structure on R 1 cross R 2 and you have the projection maps, the said theoretic projection maps P 1.

So, let us draw it like this, always prefer diagrams while doing category theory. It is easier to see what is happening be the said theoretic projection maps and those you can easily check are ring homomorphisms, because addition and multiplication are defined component wise and because the unit of R 1 cross R 2 has each component in the unit of R 1 and the unit of R 2.

So, now what we know is, so now we need to prove the universal property for a product that is the definition of a product in the category of rings. So, given a ring R and f 1 R to R 1, f2 R to R 2 we need to show that there exists so let me draw the diagram that we are after.

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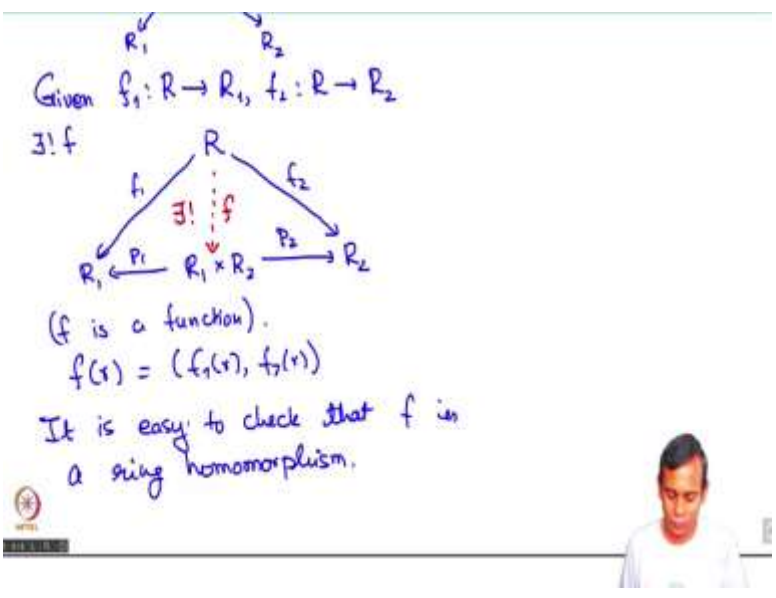

So, we have R 1 cross R 2. We have P 2 down to R 2. We have P 1 down to R 1 and we have this thing R and we are given f 1 and we are given f 2. So, now this R 1 cross R 2 is a said theoretic product, because it is a said theoretic product we know that there exist a unique function now yet a ring homomorphism, a function, there exists a unique function f. from R to R 1 cross R 2, which makes this diagram commute.

So, there exists a unique f but f is only a function so far and now all we need to do is to check that f is a ring homomorphism, but remember f is defined by f of r is equal to f 1 r comma f 2 r. And it is easy to check that f is a ring homomorphism. I will not bore you with those details. You can work them out yourself if you feel the need.

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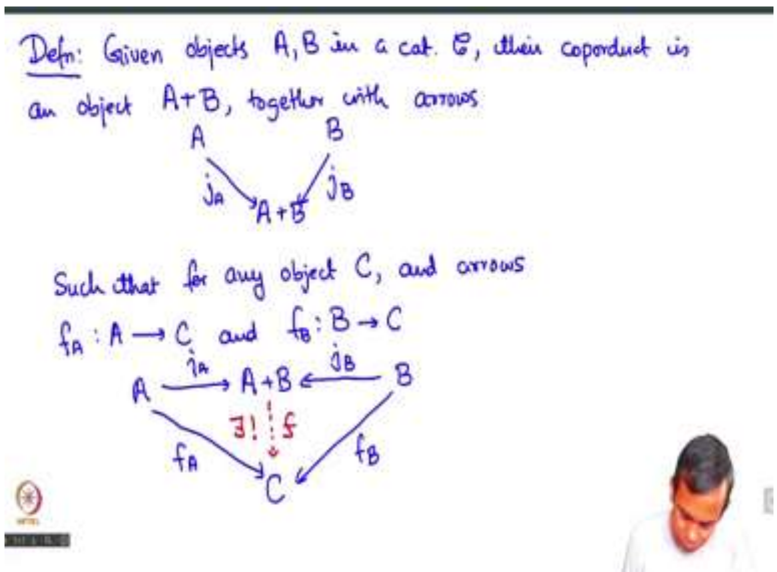

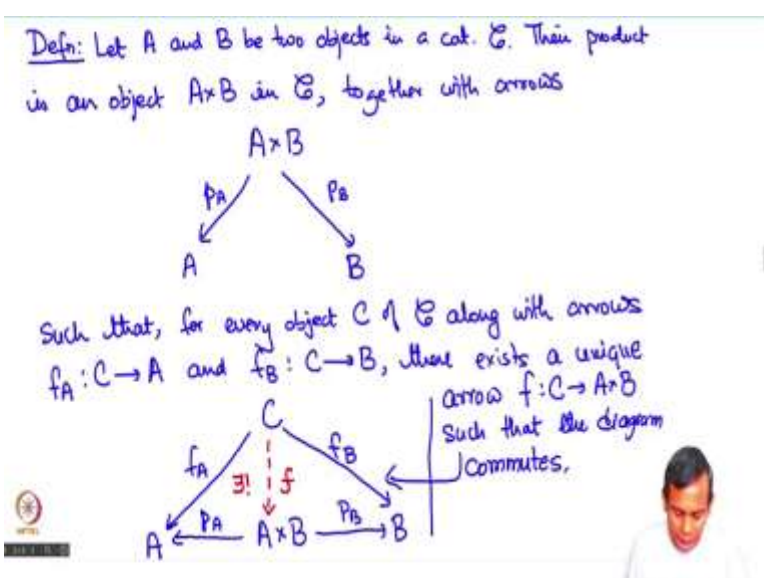

Every concept in category theory has a dual just like the dual of the concept of an initial object was a terminal object so similarly there is the dual of the notion of a product and it is called a co-product. So, let us define it and a co-product is defined as given two objects A and B in a category C their co-product is an object.

We will write A plus B together with, now the idea with dualizing in category theory is that the directions of all arrows are reversed so together with arrows and so now we have, earlier we had A and B, A cross B to A and B and now what we have is A plus B here and we have arrows from A to A plus B, which we will call j A and we have another arrow B from B to A cross B, j B such that for any object C and arrows f A from C to, A to C and f B from B to C.

So, now if you go back to the definition of a product we are reversing the directions of all the arrows here. So, earlier we had arrows from C to A and C to B. Now we have arrows from A to C and B to C. Let us draw the diagram that we want. So, it will be the reverse of the diagram we had in the definition of the product.

So, we have A plus B, we have an arrow going from A in to A plus B which is j subscript A and we have another arrow from B in to A plus B called j sub B, and we also have an object C and we have arrows, f subscript A from A to C and another arrow f subscript B from B to C. So, then what we want to say is that there exist a unique arrow which we will call f.

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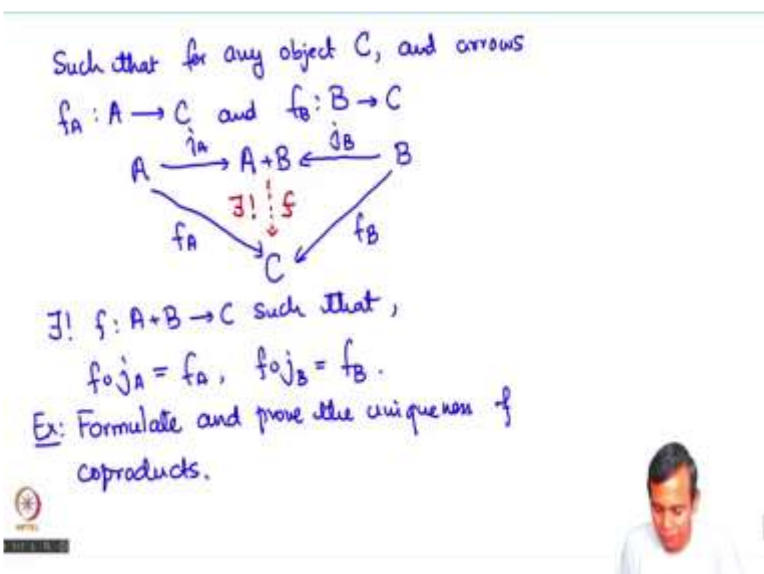

There exists a unique arrow f from A plus B to C such that the above diagram commutes which can be written in long hand as f circle j sub A equals f sub A and f circle j sub B equals f sub B. this is the definition of a co product in a category. So, it is exactly the definition of a product but with the directions of all the arrows reversed.

And now just you see if you were following what I did before with products. I will give you the following exercise. Formulate and prove the uniqueness of co-products. In general co-products or products may or may not exist but when they do exist they are unique in certain sets. So, what you have to do is you have to figure out the analog of the uniqueness statement of products for co-products.

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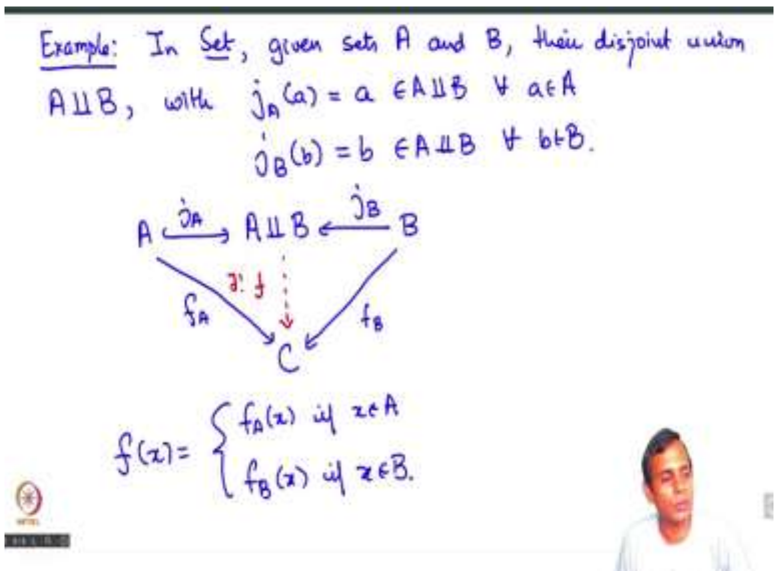

Let me give you an example of co-products. Pretty much everything in category theory we try to see what happens with the category sets. So, in the category of sets the given sets A and B their disjoint union A disjoint union B that means you will take A and B even if you have common elements in these sets A and B.

You take the common elements, you treat them as different elements so maybe you make a copy of all the elements which they share in common and may create disjoint sets, A union B. So, the carnality of the set will be the carnality of A plus the carnality of B, with j A being the inclusion map so j A of a equals a which is an element of A union B and j B of b is b which is also an element of A union B.

Note that we have this universal property so let us just check that this meets the criteria for being a co-product so we have A disjoint union B we have j A, we have j B from B and now what we are seeing is that given any set C together with functions f A and f B, now we are in the category of sets so arrows are just functions.

There exists a function f, a unique function f and indeed this function f has to be given by, so, yeah, so let us just define it like this, maybe it is better to write it like this. f of x is equal to f subscript A of x if x belongs to A and it is equal to f subscript B of x if x belongs to B. And this is the only function that will do this trick for us.

So, in the category of sets the co-product is just the disjoint union of two sets. So, you have seen the definition of products of two objects in a category and co-products of two objects in a category, but we can generalize these definitions further to arbitrary families of objects. So, products of families.

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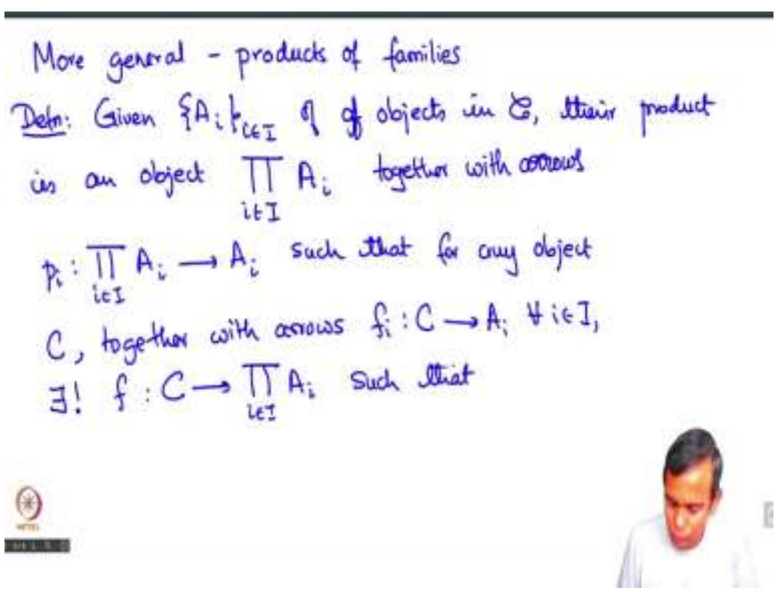

So, the definition is as follows, given a family A i i index by some set I of objects in C. Their product is an object which we denote by, I have used the pi notation, the product notation which we use by multiplying numbers and so on, together with arrows P i from product i in I A i to A i such that, and now we more or less follow the definition of a product. So, we say that whenever we have an object for any object C together with arrows f i from C to A i for each i in I there exists a unique arrow f from product A i to this product, no from C to product such that.

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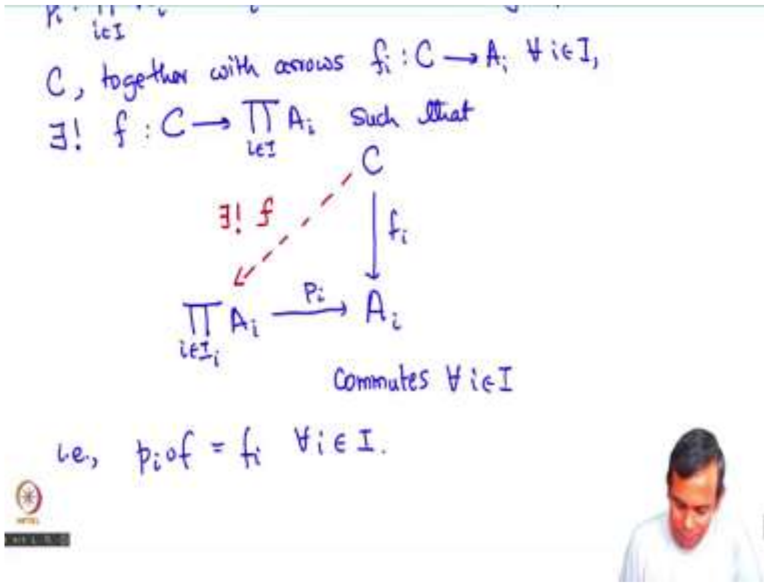

So, maybe it is better to draw a picture here. So, we have the product. i belongs to I A i and we have these projection maps. Arrows to A i and we have an object C and we are given a family of arrows from C to A i called f i and the assertion is that there exists a unique arrow from C to the product which we will call f.

This diagram commutes. In other words we are saying that P i circle f i, not P i circle f i. P i circle f is equal to f i for all I. So, we want to say for all i in this diagram commutes. For all i in I, and you can check that in the category of sets again the product is the Cartesian product of a family of objects. And similarly you can define a co-product of an arbitrary family of objects so it is the same kind of definition.

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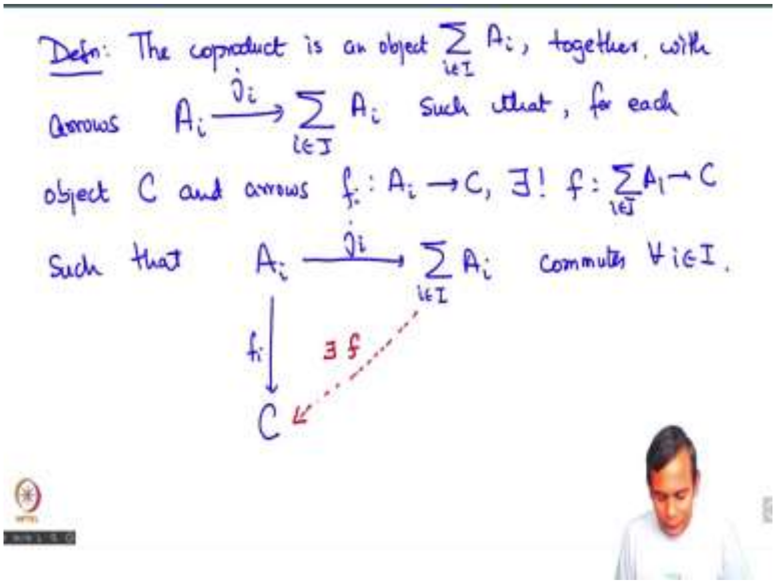

So, we usually call is summation i the co-product, same notation as before summation i in I, A i so maybe I should say is an object, together with arrows j i from each A i there is an arrow j i to this co product, i belong to I, A i, such that for every object C and arrows f i from A i to C there exists a unique arrow f from this co-product to C.

So, let me draw the diagram so we have A i and for each i we have this j i, i in I, summation A i and we have this object C and we are given, for each i we are given an arrow f i and the assertion is that there exists a unique arrow f, which makes this diagram commute for all i. And again in the category of sets, this co-product is the disjoint union of this family of sets. Let us end this lecture with a slightly different kind of example of products and co-products.

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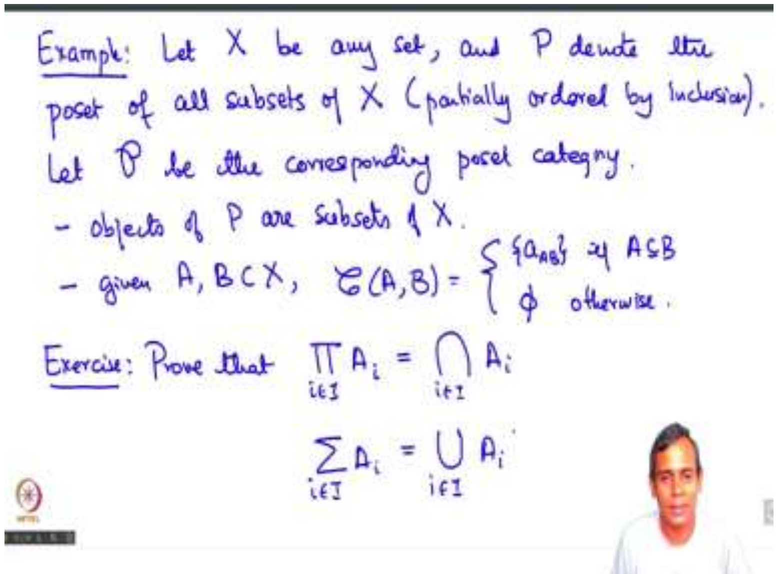

So, let X be the set, X be any set and P denote the partially ordered set of all subsets of X. Okay so partially ordered by inclusion, then we have this category script P. In this category objects are, so we have discussed this category before objects of P are in this specific case subsets of X and given subsets A B there is a unique arrow.

So, let us just write it down C A B is either, is a singleton set maybe we will call it arrow from A to B if A is contained in B and it is empty otherwise. So, this is a special case of the general category that we associate it to, the category that we associate it to, a general poset. A specialized to the case whether the poset is the set of all sub-sets of a given set X, and here is an exercise for you. Prove that in this category.

Well, we could do it in general i in I A i is just the intersection i in I A i and the co product i in I A i is just the union i in I A i. And here I do not mean disjoint union. They are just subsets of the set X. you take that union inside X. So, that is again a subset of X. So, it is just an exercise in, and if you can solve this then you would have to some extent grasped the definition of product and co-product that I gave you in this lecture.