Algebra - II
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Lecture 5
Constructible Numbers

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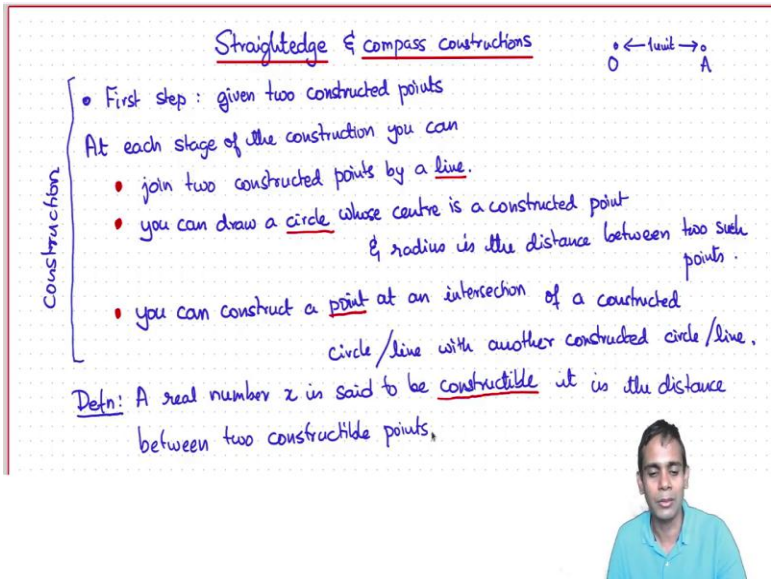

A very interesting manifestation of field extensions comes from numbers that can be constructed using a straightedge and compass. You have seen these constructions in school a straightedge is basically a ruler without any markings and a compass is a device to draw circles given a center and a radius. Now a straightedge and compass construction is a series of steps. So the first step is always that you had given two points and I will call these points constructed points.

So basically at each stage of a construction you will have a set of object that are constructed at that stage. And at each new stage you will construct a new object according to certain rules the first step you are given two constructed points. So this could just be two points and usually we denote them by O and A and the understanding is that the distance between these two points is one unit. So what can you do at each stage?

So at each stage of the construction you can do one of several things. So firstly what you can do is, you can join two constructed points by a line. When you join two constructed points by a line, then that line is among a list of constructed object. So that line has now been constructed, then

what else can you do? You can take, you can draw a circle, whose centre is a constructed point and radius is the distance between two constructed points.

So you can only draw circles with such centers at radii. And what else can you do? You can construct new points, so you can construct a point at the intersection, well there could be more than one intersection. So at an intersection of a constructed circle with or line with another constructed circle or line, constructed circle or line with another constructed circle or line. So the objects that you are constructing at this stage, if you join two constructed points say by a line, this line is constructed.

And if you draw circle whose center is constructed point and radius is the distance between two such points then the circle has been constructed. And you can construct a new point by taking the intersection of either two lines or two circles, well if you take two circles you can have two intersection points you can choose either of them and you could also take the intersection of two points. And now the definition of a constructible number is the following.

So this whole definition is the definition of a construction. It is a series of such steps, so the definition is a real number x is said to be constructible, if it is the distance between two constructible points. And you can ask which real numbers are constructible and that is the question we are going to answer. Now note that a construction is a finite sequence of steps and at each step you only have a finite set of possibilities.

So the number of constructions is countable and so also the number of point set you would have in any construction is finite. So the number of possible distances that you can, real number that you can construct is countable and so there are very few countable real numbers most real numbers are not constructible. Let me illustrate that the square root of 2 is a constructible real number it is probably a very one of the simplest non trivial examples.

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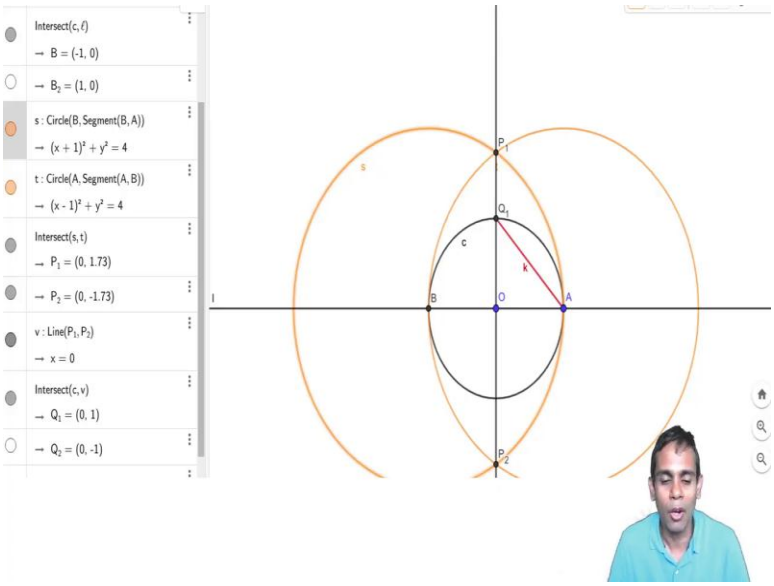

I am going to use a package called GeoGebra to do this. It very nicely implements constructions. So we start with two points O and A and the distance between these two points is (()) (7:17) to be 1 unit. And then we can draw a lines through it and we can draw circle with center O and radius given by the distance from O to A. And we call that circle c. So you can read the code here it is a circle c with center O and radius is the segment O to A.

And now we can try to find the intersection of this circle c with the line l. There are two points in intersection this point B and the other point is A. So GeoGebra gives me both the points, I will not display this other point B2 which is also the same as A. And now what we will do is we will construct a perpendicular to this line l using these point B and A. So now what I will do is I will take a circle with center B and radius B comma A, that is the orange circle I am calling it s.

And similarly I will take a circle with center A and radius the distance from A to B. I am calling that circle t. Now this circle s and t they intersect in two points P1 and P2. And so I can now these points have also been constructed and now I construct the line v by as the unique line that passes through P1 and P2. And now let us look at the intersection of v with the circle c which we are drawn earlier.

And I am calling that point Q1 there is another point of intersection which is called Q2. But I would not be really needing that and now I have this point Q1. Now because O A and O Q1 are both radii of a circle O A and O Q1 have the same length. So O Q1 is also 1 unit and by our construction this angle O A Q1 and B O Q1 are equal and so they are right angles. And so this O Q1 A is right angle triangle with right angle at the vertex O.

And so if you take the segment Q1 A, that segment has length square root 2. So this is a construction of square root 2, the rules are a little stricter than what we did in high school geometry where we were usually allow to choose any radius that we like and we also allowed to choose arbitrary points but here we are only allowed to take points which are already constructed and not introduce any new points. So the question is which numbers, how can we tell if a number is constructible or not?